

Bank Risk Management and the Franchise Value*

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First draft: October 17, 2002

This draft: February 11, 2003

Abstract

We analyze optimal risk management strategies for a regulatory restricted bank financed with deposits and equity in an infinite horizon model. The bank has a positive franchise value from rents coming from deposit related services (liquidity provision, payment services, safety storage), and from rents on the asset side due to its delegated monitoring activity. This franchise value and the liquidation costs in case of a bank run give the bank a motivation for risk management. The franchise value reduces asset substitution incentives and enhances the value of hedging strategies with a zero probability of bankrun.

JEL Classification: G1; G21; G28

Keywords: Bank; risk management; corporate hedging; franchise value

*We wish to thank the RiskLab (Zürich) for the financial support of the first author as well as for the National Centre of Competence in Research 'Financial Valuation and Risk Management' (NCCR FINRISK); NCCR FINRISK is a research programme by the Swiss National Science Foundation. We wish to thank ECOFIN Research and Consulting for the financial support of the second author. This work is part of the RiskLab project titled 'Banks' Optimal Hedging Decisions Under the Threat of Bank Runs'. We gratefully acknowledge helpful comments from Rajna Gibson.

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1 Introduction

The very existence of banks is tied to the fact that transaction costs and asymmetric information hinder direct interactions between lenders and borrowers. Banks have proven to be efficient institutions to overcome these problems, yet they do not fit into the classic microeconomic framework of perfect markets. In the banking literature (e.g. Freixas and Rochet (1998) summarize the literature), it is usually assumed that they have monopolistic power to some degree. This allows the banks to generate a net profit from deposit-related services such as liquidity provision and payment services. Moreover, their role as delegated monitors (Diamond, 1984) and their corresponding informational advantage may allow them to generate rents on the asset side as well. The present value of such rents is called franchise value.

If a bank is disintermediated due to a bank run, the overall loss thus not only comprises asset liquidation costs as in Diamond and Rajan (2000, 2001). The bank also loses the stream of future net profits from deposit-related services, as well as the monitoring rents from its credit activity. Hence, the franchise value should be a powerful incentive to reduce asset substitution or to avoid the possibility of bank runs completely.¹ This has previously been recognized in the literature. Bhattacharya et al. (2002) highlight the impact of depository rents on banks' risk taking. Demsetz et al. (1996) focus on rents on the asset side, originating from monitoring activities and long term relationships with borrowers. Their empirical results show that banks with some degree of monopolistic power or with valuable lending relationships operate more safely. Furthermore, banks with a high franchise value hold more capital and take on less portfolio risk. In a recent contribution, Pelizzon (2001) presented a multiperiod model of bank portfolio management, where the bank has a positive franchise value

¹As will become clear in the analysis of the paper, this is only the case when regulatory restrictions set upper limits on asset risk. Otherwise, banks would take a vast amount of risk which compensates the expected loss in case of a bank run.

and is restricted by regulatory constraints. The focus of Pelizzon (2001) is on the influence of the franchise value on the bank's risk taking behaviour in its investment activities. The impact of several sources of the franchise value on optimal investment decisions is analyzed.

In this paper, we examine the impact of the franchise value on a bank's hedging decision. It is assumed that the investment decision to be given. We extend a model for optimal bank risk management presented in Bauer and Ryser (2003). In this paper, we found that for a bank bank that is financed with deposits and which is subject both to regulatory restrictions and liquidation costs in case of a bank run, the well known results on hedging from corporate finance theory apply partially at best. The bank's equity is not equivalent to a call option, i.e., its value does not always increase when the volatility of the asset increases, since this also raises the likelihood of a bank run. As regulatory restrictions limit the maximum achievable risk, maximizing the assets' volatility is not optimal for shareholders whenever their expected liquidation costs of a bank run can not be outweighed by an increase in the expected return. Furthermore, the complete hedging result of Froot and Stein (1998) holds only in special cases.²

The model in Bauer and Ryser (2003) consists of one period only. Future rents have no impact on the optimal hedging decision during the single period, as the rent during the initial period is earned with certainty. To analyze the impact of future rents on the hedging decision, we extend this model to an infinite horizon setting. Our bank has a positive franchise value generated by assets and deposits. However, it can realize income from deposit-related services and its monitoring activity only as long as it is not disintermediated by a bank run. We find that asset substitution incentives are reduced not only by liquidation costs of assets, but also by a high franchise value of the bank. The higher the

²There is even a constellation where the hedging decision is shown to be irrelevant, which coincides with the result of the Modigliani-Miller-theorem. This, however, is only a special situation, where the risk management restrictions, the size of the liquidation costs in case of a bank run, and the initial debt ratio are all set in such a way that both a bank run and risk-shifting to depositors are impossible.

franchise value, the more likely the bank is to choose a safe hedging strategy for which in no state of the world a bank run occurs. This holds for all banks except for those that are already in a critical situation today and gamble for resurrection. Furthermore, regulatory constraints such as risk-based capital standards are indeed useful to reduce asset substitution incentives, since they limit the upside-potential of risk taking strategies. Especially for banks in highly competitive environments, where the franchise value is small, tighter restrictions prevent a risk maximizing strategy.

The remainder of this paper is organized as follows. In section 2, we present an overview of the model in Bauer and Ryser (2003). In section 3, we extend this model to an infinite horizon to analyze the impact of the franchise value on the hedging decision of the bank. All proofs can be found in the Appendix. Section 4 concludes.

2 Review of the one-period model of bank risk management

2.1 The market and the bank

The basic model of Bauer and Ryser (2003)³ incorporates the following market structure: We assume a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega := \{\mathcal{U}, \mathcal{D}\}$, $\mathcal{F} := \{\emptyset, \{\mathcal{U}\}, \{\mathcal{D}\}, \Omega\}$ and $\mathbb{P}(\mathcal{U}) = p$. The model has one period, between time 1 and 2 and $T \equiv \{1, 2\}$ denotes the set of time indices. The market consists of two assets: A riskless asset whose value at time 1 is normalized to 1, $B_1 = 1$, and $B_2 = B_1 R$ where $R > 1$ is some constant. Furthermore, there is a risky asset

³For the reader's convenience, we provide a summary of the model and the results presented in Bauer and Ryser (2003).

with value $P_1 > 0$ at time 1 and a value $P_2(\omega)$ at time 2 where

$$P_2(\omega) = \begin{cases} P_u \equiv P_1 u, & \omega = \mathcal{U}, \\ P_d \equiv P_1 d, & \omega = \mathcal{D}, \end{cases}$$

and we assume that

$$u > R > d. \quad (1)$$

This market, consisting of two assets with linearly independent payoffs and two states of the world, is complete. We define the unique risk-neutral probability measure Q by $Q(\mathcal{U}) = q$, for which $\mathbb{E}^Q \left[\frac{P_2}{B_2} \right] = P_1$, where $q \equiv \frac{R-d}{u-d}$. For hedging purposes, we further introduce a redundant forward contract written on the risky asset: It is entered into at time 1 at no cost, and the buyer of the contract has to buy one unit of the risky asset at time 2 at the forward price RP_1 ⁴. Hence, the value f_t of the forward contract is

$$f_1 = 0, \\ f_2(\omega) = \begin{cases} f_u \equiv P_u - RP_1, & \omega = \mathcal{U}, \\ f_d \equiv P_d - RP_1, & \omega = \mathcal{D}. \end{cases}$$

At time 1, the bank has a loan portfolio which has the same dynamics as the risky asset. Its value at $t = 1$ equals αP_1 , which corresponds to a prior position of $\alpha > 0$ units of the risky asset. Further, the bank has two sources of capital: Depository debt and equity where the latter has limited liability. The initial amount of depository debt D_1 is given. Also the deposit rate R_D is given and we assume $R_D > 1$. Hence, the nominal amount of deposits at time 2 equals $D_2 = D_1 R_D$. Since the capital structure decision is already set, shareholders can behave strategically and expropriate wealth from depositors as in Jensen

⁴We will call both long and short position in this forward ‘hedging positions’. We thus use the term ‘hedging’ for any activity that changes the probability distribution of future firm value.

and Meckling (1976). Therefore, the bank's objective is to maximize the value of the shareholders' claim.

Deposits are subject to bank runs. In run situations, assets need to be sold in fire sales in order to pay out the depositors. This creates liquidation costs (indirect bankruptcy costs) (Diamond and Rajan, 2001) since asset market prices can drastically decline if big blocks of proprietary assets have to be sold immediately. We assume liquidation costs of γV_2 , $0 < \gamma < 1$, since the bank has to sell all of its assets at once during a run. Such a bank run takes place whenever the value of the bank's assets is not sufficient to repay every depositor's full claim. In this situation, all fully informed rational depositors run to the bank at the same time causing a so-called information-based bank run (Jacklin and Bhattacharya, 1988). We have shown in Bauer and Ryser (2003) that the critical asset value below which a bank run takes places equals

$$V_L = \frac{D_2}{1 - \gamma}.$$

Without the threat of a bank run, the payoff function for the bank's equity in $t = 2$ is

$$S(V_2, D_2) \equiv \begin{cases} V_2 - D_2, & V_2 \geq D_2, \\ 0, & 0 \leq V_2 < D_2, \end{cases}$$

which is the payoff of an ordinary call option on the firm value with strike price D_2 . However, in the presence of liquidation costs, a bank run will always take place when $V_2 < V_L$, so the residual payoff to shareholders drops to zero below V_L . Since $V_L > D_2$, the equity payoff changes to

$$S(V_2, D_2) \equiv \begin{cases} V_2 - D_2, & V_2 \geq V_L, \\ 0, & 0 \leq V_2 < V_L. \end{cases} \quad (2)$$

At time 1, the bank chooses a hedging position consisting of h units of the

forward contract on the risky asset. To a chosen position h corresponds a stochastic value of the assets, $V_2(h)$, at time 2.⁵ Regulatory restrictions⁶ (lower and upper bounds) constrain the bank's choice of the hedging positions,

$$-\alpha \leq h \leq a_1, \quad (3)$$

where the lower bound $-\alpha$ is equivalent to a (no net) shortsales constraint. The upper bound $a_1 > -\alpha$ can be interpreted as a risk-based capital restriction, where the equity capital is considered fixed in the short run. Based on the existing equity capital, the bank is restricted in its risk taking. The bank's management's goal is to maximize the present value of the equity at time 1, by choosing a hedging position h subject to the given regulatory constraints. It solves the following optimization problem:

$$\max_{-\alpha \leq h \leq a_1} I(h) = \max_{-\alpha \leq h \leq a_1} \mathbb{E}^Q \left[\frac{S(V_2(h), D_2)}{B_2} \right] \quad (4)$$

where $I(h)$ denotes the objective function, i.e., the value of equity at time 1 as a function of hedging portfolio h .

Bauer and Ryser (2003) find the following three candidates for the optimal hedging decision:

- $h = a_1$ is the strategy of maximum speculation;
- $h = -\alpha$ is the case of complete hedging where the bank sells forward its whole initial position;
- In some cases the bank is indifferent between the hedging strategies in a whole range, including the complete hedging decision $h = -\alpha$; for the hedging strategies in this range, shareholders receive a positive payoff in both states of the world, and the expected value of the equity is constant.

⁵ $V_2(h) = \alpha P_2 + h f_2$.

⁶These restrictions may as well be imposed by rating agencies on the bank if it pursues the goal of keeping its current rating.

The choice of the optimal hedging position depends, among other things, on whether the asset value $\bar{V} \equiv \alpha P_1 R$ of the ‘fully hedged bank’ at time 2 (i.e., the value attained if the bank sells forward its whole position α in the risky asset and the future value becomes certain) is higher or lower than the bank run trigger V_L .

We consider first the case where the asset value of the ‘fully hedged bank’ is lower than the bank run trigger, $\bar{V} < V_L$. In this case, the payoff to shareholders would be zero if the bank hedged completely. If a positive payoff to shareholders in state \mathcal{U} is attainable with this hedging decision, a_1 is the optimal hedging decision. This is a situation of a ‘gamble for resurrection’ where it is always optimal to take as much risk as possible.

In the other case where $\bar{V} \geq V_L$, the payoff to shareholders would be positive if the bank hedged completely.

In the case where shareholders would receive a positive payoff if the bank hedged completely, $V_L < \bar{V}$ (i.e., $D_1/\alpha P_1 < (R/R_D)(1-\gamma)$) the optimal hedging decision depends on the amount of risk which is tolerated by the regulator:

- If the regulatory restrictions are loose enough, such that the maximum admissible hedging position a_1 yields a higher expected equity payoff than the ‘fully hedged’ position $-\alpha$, then a_1 is the optimal hedging decision. Since shareholders could lock in a sure positive payoff by hedging completely, this is not a ‘gamble for resurrection’, although the optimal hedging strategy is the same. Because of their nonlinear payoff, they can expropriate wealth from depositors by taking on more risk, since the increase of the payoff in state \mathcal{U} overcompensates the liquidation costs in state \mathcal{D} . This strategy is known as ‘asset substitution’ (Jensen and Meckling, 1976). The loose regulatory restrictions (a_1 large) allow for such excessive risk taking that the disciplinary effect of the bank run threat is not binding.
- If the regulatory restrictions are more constraining, such that at the max-

imum admissible position a_1 the gain in expected return would not outweigh the expected liquidation costs of this portfolio, then there is no unique optimal hedging decision. Rather, hedging strategies in a whole range including $-\alpha$ are optimal. For these hedging strategies, shareholders receive a positive payoff in both states of the world. If the initial debt ratio $\frac{D_1}{\alpha P_1}$ is higher than $\frac{d}{R_D}(1 - \gamma)$, the optimal hedging strategy is risk reducing, i.e., $h^* < 0$. Thus, banks with tight regulatory restrictions, high initial debt ratio, high asset volatility and/or high liquidation costs reduce their risk with hedging.

- In the case where the regulatory constraints are so tight that they do not admit any hedging positions that could lead to a bank run in one state of the world, there is again no unique optimal hedging decision. Rather a whole range of hedging positions including $-\alpha$ is optimal. In this case, the Modigliani-Miller-theorem of hedging-irrelevance also holds ex post, i.e., after determination of the capital structure. Shareholders are indifferent with respect to all admissible hedging strategies. This, however, is only a special case where regulatory restrictions prevent asset substitution completely since they guarantee that banks can never fall below the bank run trigger.⁷

Overall, neither the asset substitution strategy nor the complete hedging are always optimal strategies. The shareholders may have some possibility to expropriate wealth from depositors, but they incur a risk as well, namely the liquidation costs in case of a bank run. Due to regulatory restrictions on the hedging strategy, there are situations in which risk reduction enhances equity value: Whenever the expected liquidation costs are higher than the expected

⁷In another special case the value of the fully hedged bank exactly equals the bank run trigger, $V_L = \bar{V}$. Then there are only two extreme solutions. The optimal hedging strategy jumps from the complete hedging position $h^* = -\alpha$ to the maximum admissible risky position a_1 as soon as the expected liquidation costs are outweighed by the higher expected return and vice versa. In this case, only the size of the regulatory constraint a_1 determines which part of this ‘bang-bang’ strategy is optimal.

return at the maximum admissible risky position, taking excessive risk is not optimal.

3 Infinite horizon model

In the previous one-period model, the franchise value was only earned from time 1 to 2. This profit was not affected by a potential default of the bank at time 2. To capture the impact of the franchise value on the bank's risk taking, we extend the model to an infinite horizon setting, much in the spirit of Pelizzon (2001).

Pelizzon (2001) analyzes a bank, that has a positive franchise value and is subject to regulatory constraints, in a multiperiod model. Deposit insurance prevents bank runs. The bank controls its risks through its investment policy. The focus of Pelizzon's paper is the impact of various sources of rents on the bank's investment decision and on the value of the deposit insurance liability.

In our paper, we assume the bank to have fixed investments and capital structure, but it can hedge its risk by choosing a position in forward contracts. Over its lifetime, the bank has a net profit stream from deposit-related services (liquidity provision, payment services), that comes in as a cut-off on the risk-adjusted deposit-rate determined by the market.⁸ The present value of this net profit stream is called franchise value of the deposits. On the asset side, the bank holds a nontraded asset which has the same dynamics as the traded risky asset. The bank can thus hedge perfectly. However, the bank receives an additional net profit on its proprietary asset in each period. We call the present value of this profit franchise value of the assets. Both franchise values are lost in the event of a bank run. This loss constitutes, together with the liquidation costs, the bankruptcy costs which create an incentive for the bank to pursue a

⁸Since we do not model the capital structure decision, this rate is exogenously given in our model.

risk management policy.

3.1 The market

We assume that the bank has no given end date of operation. Only bank run can lead to bankruptcy and the closure of the bank. Hence, the model has infinitely many periods, i.e., $t = 1, 2, 3, \dots$. We extend the market in a straightforward way. In each period, the riskless asset has a riskless rate of return of $R > 1$, and therefore

$$\begin{aligned} B_1 &= 1, \\ B_t &= B_{t-1}R, \quad t = 2, 3, \dots \end{aligned}$$

The return on the risky asset is given by a sequence of independent and identically distributed random variables which have the same distribution as the return in the one-period model in section 2. We now have a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ which is the product space of $(\Omega, \mathcal{F}, \mathbb{P})$.⁹ Thus, an element $\tilde{\omega} \in \tilde{\Omega}$ is of the form $\tilde{\omega} = (\omega_1, \omega_2, \dots)$, where $\omega_i \in \Omega$, $i = 1, 2, \dots$. We denote by $\tilde{\omega}_t \equiv (\omega_1, \dots, \omega_t)$ the projection onto the first t coordinates. By ω_t we denote the event at time t .¹⁰

The risky asset has a value $P_1 > 0$ at time 1, and independent and identically distributed returns, i.e., its price has the dynamics

$$P_{t+1}(\tilde{\omega}) = \begin{cases} P_t(\tilde{\omega}_t)u, & \omega_{t+1} = \mathcal{U}, \\ P_t(\tilde{\omega}_t)d, & \omega_{t+1} = \mathcal{D} \end{cases}$$

for $t \geq 2$. The assumption

$$u > R > 1 > d$$

⁹Let $\tilde{\Omega} \equiv \Omega^\infty$ be the space of all Ω valued sequences. The projection onto the t -th coordinate $p_t : \tilde{\Omega} \rightarrow \Omega$ is the mapping $\tilde{\omega} \mapsto \omega_t$. Then, \tilde{F} is defined as $\tilde{F} \equiv \sigma(p_t, t = 1, 2, \dots)$. The probability measure $\tilde{\mathbb{P}}$ is the product measure of \mathbb{P} (Bauer, 1996).

¹⁰More precisely, $\omega_t(\tilde{\omega}) \equiv p_t(\tilde{\omega})$, but we will not refer to $\tilde{\omega}$ in the following.

is maintained. The one-period risk-neutral probability measure is extended to the product measure \tilde{Q} . In each period, there is a one-period forward contract on the risky asset available for hedging purposes.¹¹ The time t forward contract is entered into at time t at no cost, and the buyer of the contract has to buy one unit of the risky asset at time $t + 1$ at the forward price RP_t . Hence, the value $f_s^{(t)}$ of the time t forward contract (the superscript denotes the time when the forward contract is entered into) is

$$f_s^{(t)} = 0, s \neq t + 1$$

$$f_{t+1}^{(t)}(\tilde{\omega}) = \begin{cases} P_t(u - R), & \omega_{t+1} = \mathcal{U}, \\ P_t(d - R), & \omega_{t+1} = \mathcal{D}. \end{cases}$$

3.2 The bank

At time 1, the bank has a loan portfolio which has the same dynamics as the risky asset. Its initial value is $v > 0$. In every period, the bank can hold a position of h_t units in the one-period forward contract on the traded asset. It will be convenient to focus on the quantity $H_t \equiv h_t P_t$ which plays the following role: If a position of h_t units is taken in the forward contract $f_t^{(t)}$, this position will have a value at time $t + 1$ of $h_t P_t(u - R)$ in state \mathcal{U} and $h_t P_t(d - R)$ in state \mathcal{D} . The two possible outcomes can then be written in the form $H_t(u - R)$ and $H_t(d - R)$. H_t is the notional value of the hedging position. We call any sequence $\tilde{H} \equiv \{H_t\}_{t \geq 1}$ of positions in the forward contract a hedging strategy and denote by $H^t \equiv (H_1, \dots, H_t)$ the hedging decisions up to time t ('hedging history up to time t ').

To each hedging strategy \tilde{H} , we associate a sequence of stochastic firm values $V_t(H^{t-1})$, $t = 1, 2, \dots$, conditional on the hedging decisions up to time $t - 1$. During each period, depositors receive interest on their money. The accrued

¹¹As we will show below, the decision problem of the bank is independent of time. Hence, the assumption of a one-period forward contract is not unnecessarily constraining.

interest on the deposits is paid out at the end of each period, and deposits are rolled over unless there is a bank run. Therefore, the value of the deposits after payment of interest is constant. Before interest payments, the nominal value of the deposits increases from the initial volume D to DR_D where $R_D > 1$ is the exogenously given deposit rate. After interest payment, the value of the deposits is constant,

$$D_t = D, t = 1, 2, \dots$$

By $V_L = \frac{DR_D}{1-\gamma}$, we denote again the firm value below which a bank run is triggered: If $V_{t+1}(H^t) < V_L$, a bank run takes place and all assets are sold to pay out the deposit holders.¹² The firm value V_s then is equal to zero for all future times $s \geq t$.

To keep the model analytically tractable, we assume – similar to Pelizzon (2001) – that the solvent bank attempts to keep a target capital structure. As will become clear below, this assumption lets the conditional distribution of the firm value become constant with respect to time which facilitates the problem considerably. Since we assume that the capital structure is given exogenously, the bank’s management just maintains the initial capital structure. The target equity capital is expressed as a fraction of deposits, kD , $k > 0$, where the target equity capital is the equity available at time 1,

$$v = (1 + k)D.$$

At each time $t = 2, 3, \dots$ when the bank is solvent, a capital payment is exchanged between the bank and its shareholders. If the bank’s capital exceeds the target capital, shareholders receive a dividend. Otherwise, if the capital is below the target capital but the bank is still solvent, shareholders pay in new capital. We call this capital payment, which has either a positive or a negative

¹²If this does not happen until time t , we say that the bank has not defaulted or is solvent at time t .

sign, dividend. The dividend payment $d_t(H^{t-1})$ at time $t = 2, 3, \dots$, conditional on the fact that the bank has not defaulted by time $t - 1$, is given by

$$d_t(H^{t-1}) \equiv \begin{cases} V_t(H^{t-1}) - R_D D - kD, & V_t(H^{t-1}) \geq V_L, \\ 0, & 0 \leq V_t(H^{t-1}) < V_L. \end{cases} \quad (5)$$

Given that the bank has defaulted before time t , the dividend d_t is zero. If the bank is solvent at time $t - 1$ and does not default between times $t - 1$ and t , the dividend amounts to the difference between the actual equity $V_t(H_{t-1}) - R_D D$ and the target equity capital kD . Therefore, the firm value after dividend and interest payment, conditional on the fact that the bank has not defaulted by time $t - 1$, can take on only two values. In the event of a bank run the firm value is zero. In the case of no bank run at time t , $V_t(H^{t-1}) \geq V_L$, it is

$$V_t(H^{t-1}) - d_t(H^{t-1}) - (R_D - 1)D = v.$$

Hence, the firm value after dividends and interest payment always equals v since any surplus equity is paid to shareholders or missing equity is raised from shareholders. Conditional on the fact that the bank has not defaulted by time t , the distribution of the firm value $V_{t+1}(H^t)$ at time $t + 1$ then only depends on the hedging decision made at time t , corresponding to $H_t = h_t P_t$,

$$V_{t+1}(H^t, \tilde{\omega}) \equiv \begin{cases} vu + H_t(u - R), & \omega_{t+1} = \mathcal{U}, \\ vd + H_t(d - R), & \omega_{t+1} = \mathcal{D}. \end{cases} \quad (6)$$

Hence, the assumed dividend policy and the assumption of independent and identically distributed returns imply that the conditional distribution of the firm value is time invariant. We will express the firm value, by a slight abuse of notation, as a function of the current hedging decision only, $V_{t+1}(H_t)$. The

following notation will also be useful,

$$V_u(H_t) \equiv vu + H_t(u - R), \quad (7)$$

$$V_d(H_t) \equiv vd + H_t(d - R). \quad (8)$$

Since the dividend at time t only depends on the firm value at time t , it is also only affected by the hedging decision at time $t - 1$. Therefore, (5) becomes

$$d_t(H^{t-1}) \equiv \begin{cases} V_t(H_{t-1}) - R_D D - kD, & V_t(H_{t-1}) \geq V_L, \\ 0, & 0 < V_t(H_{t-1}) < V_L. \end{cases} \quad (9)$$

Hence, we will express the dividend as a function of the current hedging decision only, $d_t(H_{t-1})$. We will use the state variable \mathcal{S}_t to indicate whether the bank is solvent at time t or, otherwise, has experienced a bank run until time t :

$$\mathcal{S}_t = \begin{cases} 1, & \text{if the bank is solvent at time } t, \\ 0, & \text{if the bank is not solvent at time } t. \end{cases}$$

Conditional on solvency at time $t - 1$, $\mathcal{S}_{t-1} = 1$, the state variable \mathcal{S}_t takes on the following values:

$$\mathcal{S}_t = \begin{cases} 1, & V_t(H_{t-1}) \geq V_L, \\ 0, & V_t(H_{t-1}) < V_L. \end{cases}$$

For simplicity, we will also express solvency at time t as a function of the previous hedging decision, $\mathcal{S}_t(H_{t-1})$. Being a function of firm value V_t only, $\{\mathcal{S}_t\}$ is also a Markovian process with time invariant conditional distribution.

As in the one-period model, the regulatory constraint $A_1 > -v$ sets an upper limit on the admissible hedging positions:¹³ \tilde{H} is feasible if $-v \leq H_t \leq A_1$ for

¹³We also impose a (no net) shortsales constraint on the bank's hedging decision. The bank may not hold positions smaller than $-v$. The hedging position of $-v$ corresponds to complete hedging where the bank sells forward its whole risk. Since the bank cannot be net short on the asset side, we can omit cases where $V_u(h) < V_d(h)$.

each $t = 1, 2, \dots$. The set of feasible hedging strategies is denoted by Γ .

Whenever the bank has not defaulted by time t (hence, $\mathcal{S}_t = 1$) and can continue its business for a further period, it receives a net profit from its deposit-related services as well as a net profit from its proprietary asset¹⁴. We assume that the rent earned from deposits in a single period is proportional to the volume of deposits, amounting to $\tau_D D$ where $0 < \tau_D < 1$. Similarly, we assume that the rent earned from the proprietary asset is proportional to the value of the asset, amounting to $\tau_A v$ where $0 < \tau_A < 1$. The total rent $\varphi(\mathcal{S}_t)$ earned during one period thus equals

$$\varphi(\mathcal{S}_t) \equiv \begin{cases} \tau_D D + \tau_A v, & \text{if } \mathcal{S}_t = 1, \\ 0, & \text{otherwise.} \end{cases}$$

It will be convenient to summarize the rent in one parameter, called ‘aggregate rent’,

$$\tau \equiv \tau_D + \tau_A(1 + k) \tag{10}$$

such that the rent that the liquid bank earns in one period can be written as $\varphi(1) = \tau_D D + \tau_A v = (\tau_D + \tau_A(1 + k))D = \tau D$.

3.3 The optimization problem

We identify the hedging positions for which no bank run takes place in the following

Lemma 1. *Given that the bank has not defaulted by time t (i.e. $\mathcal{S}_t = 1$), there is no no bank run at time $t + 1$ for $\omega_{t+1} = \mathcal{U}$ for hedging positions*

$$H \geq \frac{V_L - vu}{u - R} =: \tilde{K}^u; \tag{11}$$

¹⁴We will call both of these profits ‘rents’.

there is no bank run at time $t + 1$ for $\omega_{t+1} = \mathcal{D}$ for hedging positions

$$H \leq \frac{V_L - vd}{d - R} =: \tilde{K}^d. \quad (12)$$

Further, $\tilde{K}^d \leq \tilde{K}^u$ if and only if $vR \leq V_L$.

This Lemma characterizes the hedging positions by which the bank can prevent bank runs. \tilde{K}^u is the smallest hedging position for which the total value of the bank's assets is sufficiently high to prevent depositors from running in state \mathcal{U} . Analogously, \tilde{K}^d is the largest hedging position for which the total value of the bank's assets is just sufficient to prevent a bank run in state \mathcal{D} .

If the initial debt ratio is sufficiently low, i.e., $vR \geq V_L$, then $\tilde{K}^u \leq \tilde{K}^d$ holds.¹⁵ For hedging positions $H \in [\tilde{K}^u, \tilde{K}^d]$, there will be no bank run state \mathcal{U} since $H \geq \tilde{K}^u$. Also, there will also be no bank run state \mathcal{D} since $H \leq \tilde{K}^d$. Hence, for hedging positions in the set $[\tilde{K}^u, \tilde{K}^d]$, there will no bank run take place in both states of the world. We call this set of hedging positions for which no bank run occurs in both states of the world the set of 'safe strategies'. Further, under the condition $vR \geq V_L$, the complete hedging decision, $H = -v$, belongs to the set of 'safe strategies', $\tilde{K}^u \leq -v \leq \tilde{K}^d$. Analysis of the complete hedging position gives the following insight on the role of the condition $vR \geq V_L$: By entering a forward position of $-v$, the bank can sell forward all of its risk. Its assets attain a future value of vR with certainty. If $vR \geq V_L$, this asset value is sufficiently high and there will be no bank run. The second case is when the initial debt ratio is that high that complete hedging yields an asset value below the bank run trigger, $vR < V_L$, and a bank run will take place with certainty. In this case $\tilde{K}^d < \tilde{K}^u$ holds and no hedging positions are available such that the bank could completely eliminate the risk of a bank run.¹⁶

¹⁵Solving the inequality $\tilde{K}^u \leq \tilde{K}^d$ for V_L immediately yields $V_L \leq vR$.

¹⁶We assume that the regulatory constraint admits risky strategies for which a bank run would take place in one state of the world. Otherwise the bank has no decision problem. This amounts to the assumption that $A_1 \geq \tilde{K}^d$ in the case where $\tilde{K}^u < \tilde{K}^d$, and $A_1 \geq \tilde{K}^u$ otherwise.

The following notation will be convenient:

$$\pi(H_t) \equiv Q(V_{t+1}(H_t) \geq V_L | \mathcal{S}_t = 1)$$

denotes the risk-neutral conditional ‘survival probability’ of the bank over the next period.¹⁷ $\pi(H_t)$ is the Q -probability that the bank will not default in the period from time t to time $t + 1$, given that it is solvent at time t and chooses the hedging position H_t . This probability does not depend on time t due to our assumption of stationary dynamics of the risky assets and that the firm value after interest payments and dividends is either v or 0 for all t . It follows from Lemma 1 that $\pi(H_t)$ takes on the following values:¹⁸

$$\pi(H_t) = \begin{cases} 1, & \tilde{K}^u \leq H_t \leq \tilde{K}^d, \\ q, & \tilde{K}^u \leq H_t, \tilde{K}^d < H_t, \\ 0, & \text{otherwise.} \end{cases} \quad (13)$$

Maximizing shareholder value in the infinite horizon model is equivalent to maximizing the expected value of the sum of discounted dividends under the risk-

¹⁷For the sake of brevity, will call $\pi(\cdot)$ the survival probability, without stressin that it is in fact a conditional probability.

¹⁸As stated in footnote 13, the case where $V_u(h) < V_d(h)$, or equivalently $\pi(H_t) = 1 - q$, can be omitted for feasible strategies.

neutral measure \tilde{Q} , so the objective function at $t = 1$ becomes¹⁹

$$\tilde{J}(\mathcal{S}_1, \tilde{H}) \equiv \varphi(\mathcal{S}_1) + \sum_{t=2}^{\infty} \mathbb{E}^{\tilde{Q}} \left[\frac{\varphi(\mathcal{S}_t(H_{t-1})) + d_t(H_{t-1})}{R^{t-1}} \right].$$

The bank's optimization problem is

$$\max_{\tilde{H} \in \Gamma} \tilde{J}(\mathcal{S}_1, \tilde{H}) := \tilde{J}(\mathcal{S}_1),$$

where $\tilde{J}(\mathcal{S}_1)$ is the value function including the rent during the initial period. Since the rent earned from deposits during the first period does not depend on the hedging decision, we consider the following objective function

$$J(\mathcal{S}_1, \tilde{H}) \equiv \sum_{t=2}^{\infty} \mathbb{E}^{\tilde{Q}} \left[\frac{\varphi(\mathcal{S}_t(H_{t-1})) + d_t(H_{t-1})}{R^{t-1}} \right].$$

We thus consider the optimization problem

$$\max_{\tilde{H} \in \Gamma} J(\mathcal{S}_1, \tilde{H}) =: J(\mathcal{S}_1),$$

where $J(\mathcal{S}_1)$ is the value function without the rent during the initial period. For this infinite horizon optimization problem, the dynamic programming equation

¹⁹We use here the following result: If $f : \tilde{\Omega} \rightarrow \mathbb{R}$ is a measurable function that depends only on the t -th coordinate ω_t , then

$$\begin{aligned} \mathbb{E}^{\tilde{Q}}[f] &= \int_{\Omega} \int_{\Omega} \dots \int_{\Omega} f(\omega_1, \omega_2, \dots, \omega_t, \omega_{t+1}, \dots) d\tilde{Q}(\omega_1, \omega_2, \dots, \omega_t, \omega_{t+1}, \dots) \\ &= \int_{\Omega} \int_{\Omega} \dots \int_{\Omega} f(\omega_1, \omega_2, \dots, \omega_t, \omega_{t+1}, \dots) dQ(\omega_1) dQ(\omega_2) \dots dQ(\omega_t) dQ(\omega_{t+1}) \dots \\ &= 1 \cdot 1 \cdot \dots \int_{\Omega} f(\omega) dQ(\omega), \end{aligned}$$

where the second equality follows from the definition of \tilde{Q} as a product measure. The distribution of the dividend d_t at time t is only affected by the uncertainty at time t , it depends only on ω_t . Also the rent earned at time t depends only on whether the bank is still solvent after time t . Therefore, $\varphi(\mathcal{S}_1) + \mathbb{E}^{\tilde{Q}} \left[\sum_{t=2}^{\infty} \frac{\varphi(\mathcal{S}_t(H_{t-1})) + d_t(H_{t-1})}{R^{t-1}} \right] = \varphi(\mathcal{S}_1) + \sum_{t=2}^{\infty} \mathbb{E}^{\tilde{Q}} \left[\frac{\varphi(\mathcal{S}_t(H_{t-1})) + d_t(H_{t-1})}{R^{t-1}} \right]$.

is as follows:

$$J(\mathcal{S}_t) = \max_{-v \leq H_t \leq A_1} \frac{1}{R} \left(\mathbb{E}^Q[\varphi(\mathcal{S}_{t+1}(H_t)) + d_{t+1}(H_t) + J(\mathcal{S}_{t+1}(H_t)) | \mathcal{S}_t] \right). \quad (14)$$

It is obvious that the value of equity is zero when the bank is insolvent, $J(0) = 0$. All stochastic quantities in (14), the state variable \mathcal{S}_t and the dividends d_t , have independent and identical distributions. Therefore, the dynamic programming equation does not depend on time and any optimal hedging policy will be stationary, $\tilde{H}^* = \{H_t^*\}_{t \geq 1}$ with $H_t^* = H^*$, $t = 1, 2, \dots$

In the remaining, we will use the parameter τ (see equation (10)) denoting the aggregate rent. The following Lemma discusses the possible candidates for the optimal hedging position H^* , i.e., positions for which the maximum of the dynamic programming equation (14) is attained.

Lemma 2. *There are two candidate sets on which the maximum of the right hand side of equation (14) can be attained:*

1. A_1 is a candidate if and only if

$$vu + A_1(u - R) + \tau D - D(R_D + k) > 0. \quad (15)$$

If A_1 is the optimal hedging position, the value of the equity of the solvent bank equals

$$\tilde{J}(1) = \pi(A_1) \frac{vu + A_1(u - R) + \tau D - D(R_D + k)}{R + (1 - \pi(A_1)) - 1} + \tau D. \quad (16)$$

2. The set $[-v, \tilde{K}^d]$ is a candidate if and only if

$$Rv + \tau D - D(R_D + k) > 0. \quad (17)$$

If the optimal hedging position H^* belongs to the set $[-v, \tilde{K}^d]$, the value of the equity of the solvent bank equals

$$\tilde{J}(1) = \pi(H^*) \frac{Rv + \tau D - D(R_D + k)}{R - 1} + \tau D. \quad (18)$$

Condition (15) says that A_1 is a candidate if the resulting dividend plus the rent earned if the bank is still solvent are positive in state \mathcal{U} . Indeed, the dividend attained with a hedging position A_1 in state \mathcal{U} is $vu + A_1(u - R) - D(R_D + k)$ (see (9) for the definition of the dividend).

Similarly, condition (17) says that the hedging positions $[-v, \tilde{K}^d]$ are candidates if the resulting expected dividend plus the rent earned if the bank is still solvent are positive. Indeed, the expected dividend for the hedging positions where there is no bank run in both states of the world ($H_t \in [-v, \tilde{K}_d]$) equals

$$\mathbb{E}^Q[V_{t+1}(H_t) - D(R_D + k)] = Rv - D(R_D + k). \quad (19)$$

The value of the equity \tilde{J} , as given in (16) and (18), consists of the following components: The rent earned during the first period, τD , which is not affected by the hedging decision, plus the present value of two perpetuities. One perpetuity is the expected future dividend stream and the other is the future stream of expected rents from assets and deposits. The applicable discount rate adjusts for the bank's default probability:

- In the case where the strategy of maximum speculation is optimal, $H^* =$

A_1 , the first term in (16) can be written as

$$\begin{aligned} & \pi(A_1) \frac{vu + A_1(u - R) - D(R_D + k) + \tau D}{R + (1 - \pi(A_1)) - 1} \\ &= \sum_{t=2}^{\infty} \pi(A_1) \frac{vu + A_1(u - R) - D(R_D + k) + \tau D}{(R + (1 - \pi(A_1)))^{t-1}} \\ &= \mathbb{E}^{\tilde{Q}} \left[\sum_{t=2}^{\infty} \frac{d_t(A_1) + \varphi(1)}{(R + (1 - \pi(A_1)))^{t-1}} \right] \end{aligned}$$

where the last equality follows from

$$\mathbb{E}^{\tilde{Q}} [d_t(A_1) + \varphi(1)] = \pi(A_1)(vu + A_1(u - R) - D(R_D + k) + \tau D).$$

Hence, the discount rate used to determine the present value of the perpetuity equals the risk-free rate R plus the risk-neutral default probability $1 - \pi(A_1)$.

- For the case where a safe hedging strategy is optimal, $H^* \in [-v, \tilde{K}^d]$, the survival probability is one, $\pi(H^*) = 1$ (see (13)), and the first term in (18) can be written as

$$\begin{aligned} \pi(H^*) \frac{Rv - D(R_D + k) + \tau D}{R - 1} &= \sum_{t=2}^{\infty} \frac{Rv - D(R_D + k) + \tau D}{R^{t-1}} \\ &= \mathbb{E}^{\tilde{Q}} \left[\sum_{t=2}^{\infty} \frac{d_t(H^*) + \varphi(1)}{R^{t-1}} \right] \end{aligned}$$

where the last equality uses (19). Hence, the discount rate equals the risk-free rate R . Due to the fact that, according to (13), for safe strategies the default probability is equal to zero, i.e., $1 - \pi(H) = 0$, the discount rate also equals the risk-free rate adjusted by the default probability.

Lemma 2 shows that the impact of the rents on the equity value is larger for safe strategies than for the risky strategy: If the value of equity $\tilde{J}(1)$ is interpreted as a function of τ , its partial derivative $\frac{\partial \tilde{J}(1)}{\partial \tau}$ in equation (16),

where A_1 is the optimal hedging position, is smaller than the respective partial derivative in equation (18), where the safe hedging strategies are optimal.²⁰

The franchise value is the difference between the actual value of the equity and its nominal value. In the case where $H^* \in [-v, \tilde{K}^d]$, it is equal to

$$\pi(H^*) \frac{Rv + \tau D - D(R_D + k)}{R - 1} + \tau D - kD,$$

whereas, for $H^* = A_1$, it is

$$\pi(A_1) \frac{vu + A_1(u - R) + \tau D - D(R_D + k)}{R + (1 - \pi(A_1)) - 1} + \tau D - kD.$$

3.4 The optimal hedging strategy

We now turn to the impact of the franchise value, as captured by the parameter τ , on the hedging decision. To this end, it is convenient to express the optimal hedging policy H^* in terms of this parameter.

Proposition 1. *If both conditions (15) and (17) hold, the bank's optimal hedging decision is as follows:*

1. *If $vR < V_L$, then $H^* = A_1$;*
2. *Else, if $vR \geq V_L$, there are three cases:*
 - (a) *If $\tau < \tau_*$, then $H^* = A_1$,*
 - (b) *If $\tau > \tau_*$, then $H^* \in [-v, \tilde{K}^d]$,*
 - (c) *If $\tau = \tau_*$, then there is no unique hedging policy, both A_1 and the set $[-v, \tilde{K}^d]$ are optimal,*

²⁰In (16), $\frac{\partial \tilde{J}(1)}{\partial \tau} = (\frac{\pi(A_1)}{R + (1 - \pi(A_1)) - 1} + 1)D$, whereas in (18), $\frac{\partial \tilde{J}(1)}{\partial \tau} = (\frac{\pi(H^*)}{R - 1} + 1)D$. For safe strategies, the survival probability equals one. From the inequality $\frac{1}{R - 1} > \frac{\pi(A_1)}{R + (1 - \pi(A_1)) - 1}$ follows that the latter derivative is higher than the former.

where

$$\tau_* \equiv \frac{q}{1-q} \frac{R-1}{R} \frac{vu + A_1(u-R)}{D} + k \left(1 - \frac{R-q}{1-q} \right) + R_D - \frac{R-q}{1-q}. \quad (20)$$

If both conditions (15) or (17) in Lemma 2 hold, A_1 and the set $[-v, \tilde{K}^d]$ are candidates. Otherwise, if any of conditions does not hold, there is either no optimal hedging decision (of both conditions are not satisfied) or only one candidate which then is optimal.

Part 1 of Proposition 1 shows that banks with a low initial amount of equity (even when they hedge completely, they face a bank run in the next period with certainty) will always hold the riskiest admissible position. By taking excessive risk, shareholders can enhance their return in one state of the world, whereas the pay-off in the other state remains zero. In the literature, this situation is known as ‘gambling for resurrection’.

Part 2 shows that when the asset value is sufficient to prevent a bank run in both states of the world by hedging completely, then the franchise value provides an incentive to choose a safe hedging strategy. The higher a bank’s franchise value of assets and deposits then is (i.e., the higher the rent τD earned in each period), the more likely it is to choose a safe hedging strategy. All banks with a high franchise value, i.e., $\tau > \tau_*$, and a high initial debt ratio ($\frac{D}{v} > \frac{d}{R_D}(1-\gamma)^{21}$ and hence $\tilde{K}^d < 0$) implement risk reducing strategies. For these banks, both the default probability and the loss given default are high. Because of given regulatory constraints, this large expected loss cannot be compensated through excessive risk taking.

Several remarks on τ_* should be made.

- Regulatory constraints on banks’ risk taking, such as risk-based capital

²¹ $\frac{D}{v} > \frac{d}{R_D}(1-\gamma)$ is equivalent to $vR > V_L$.

standards, can indeed prevent excessive risk taking. Tight restrictions, that is a low A_1 , lead to a small value of τ_* . Hence, only few banks will hold the maximum admissible risky position.

- When competition among banks rises, τ is reduced. More banks will then adopt risky hedging strategies. This is an important fact for regulators who want to control banks' overall risk exposure. The more intense competition becomes, the tighter regulatory restrictions have to be to reduce asset substitution and gambling for resurrection incentives. Empirical support in favour of this conclusion can be found in Keeley (1990).
- A higher amount of equity, i.e., an increase in k , leads to a decrease in τ_* ²² and thus induces safer hedging positions.
- An increase in the deposit rate R_D leads to an increase of τ_* and leads the bank to take taking riskier positions. This results from the increase of the debt ratio and the subsequent increase of the default probability.
- An increase in the amount of deposits D decreases τ_* , thus, safer hedging behavior is induced. This is due to our assumption that the rent earned in each period is τD , i.e., it is proportional to the volume of the deposits. With this assumption, the total franchise value of deposits increases with the volume of the deposits. In other words, an increased franchise value at stake reduces the amount of risk taken by the bank.

The results of this paper hold also in the case without liquidation costs in case of a bank run, $\gamma = 0$. In this case, depositors run for their money whenever the total value of the bank's assets falls below the nominal value of the deposits (in the literature, this is called information-based bank run (Jacklin and Bhattacharya, 1988)). The bank run trigger V_L is then equal to DR_D . All results of this paper, especially Proposition 1, apply to this case as well, since

²²Notice that $1 - \frac{R-q}{1-q} < 0$.

no additional assumption was made on the size of V_L . The bank faces as costs of a bank run the loss of its franchise value. The larger this franchise value, the greater its incentive to pursue a safe hedging strategy to avoid run the risk of facing these costs.

4 Conclusion

We have presented a model of a bank with a franchise value from its deposits and assets. We focus on the impact the franchise value has on the bank's optimal hedging decision. Since banks' deposits are a production element, they may generate a net profit on the bank's liabilities, as long as the bank has monopoly power to some extent. Furthermore, informational advantages due to delegated monitoring may allow a net profit on the asset side as well. The perpetuity of these net profit streams (the franchise value) is lost completely during a run. Besides this potential loss, the bank also faces liquidation costs on its assets in case of a bank run. Extending the one-period model in Bauer and Ryser (2003), we find that the franchise value depends on the size of the cut-off from the deposit rate, on the excess return of the bank's proprietary asset and on the probability of a bank run during a given period. This probability can be controlled by the bank's hedging decision. Compared to the one-period case in Bauer and Ryser (2003), hedging strategies with a zero bank run probability become relatively more valuable and asset substitution incentives are reduced.

The main driving factors for the determination of the optimal hedging decision are regulatory constraints and expected costs of a bank run (liquidation costs of assets and the loss of the franchise value, weighted by the probability of the bank run). Regulatory constraints limit the maximum achievable risk and force the bank to trade off asset substitution incentives against expected cost of a bank run. We find a critical level of rents that a bank has to earn in each period below which it is always optimal to take the maximum admissible risk.

Banks earning rents above this level rather choose a safe hedging strategy, in order not to lose the franchise value. If the franchise value remains fixed while regulations are loosened and thereby allow to take more risk, then the increase in expected return can be sufficient to outweigh the expected loss of the franchise value. Hence, everything else equal, more banks will hold the maximum admissible risk.

On the other hand, stronger competition among banks will reduce the franchise value and also cause more banks to hold the maximum admissible risk. This is an important fact for regulators who want to control banks' overall risk exposure. The more intense competition becomes, the tighter regulatory restrictions have to be in order to reduce asset substitution and gambling for resurrection incentives. This result is in line with the empirical examination of Keeley (1990).

Our results are based on the assumption of a constant target capital structure of the bank. It would be interesting to extend this analysis to the case where equity capital cannot always be raised to adjust the capital structure towards the target ratio. The optimal hedging decision will then probably not be stationary any more. By assuming perfect hedgeability of the banks' assets, we also ignore possible problems of non-tradeability, arising from the opaqueness of these loans to outsiders. The market would be incomplete in this case, and the determination of a unique objective function for the bank would be much more challenging. We also overlooked potential agency problems between the managers of the bank and the shareholders. These interesting topics are left open for further research.

Proofs

Proof of Lemma 1. The bank does not default at time $t + 1$ if the value of its assets is larger than the bank run trigger V_L , $V_{t+1}(H_t) \geq V_L$. Given that the bank has not

defaulted by time t (i.e. $\mathcal{S}_t = 1$), the firm value at time $t + 1$ is by (6) $V_u(H_t) = vu + H_t(u - R)$ if $\omega_{t+1} = \mathcal{U}$. Then $V_u(H_t) \geq V_L$ is equivalent to $H \geq \frac{V_L - vu}{u - R}$. The second inequality follows by a similar argument. Solving $\tilde{K}^d \leq \tilde{K}^u$ for v yields immediately $vR \leq V_L$. \square

In order to prove Lemma 2, we use the following

Lemma 3. *The value function for the solvent bank can be written as*

$$\begin{aligned} J(1) = & \max_{-v \leq H \leq A_1} \frac{1}{R} \{ q(V_u(H) - R_D D - kD) \mathbb{1}_{[V_L, \infty)}(V_u(H)) \\ & + (1 - q)(V_d(H) - R_D D - kD) \mathbb{1}_{[V_L, \infty)}(V_d(H)) \\ & + \pi(H)(\tau_D D + \tau_A v + J(1)) \}. \end{aligned} \quad (21)$$

Proof of Lemma 3. We need to analyze the elements of the dynamic programming equation (14)

$$\begin{aligned} \tilde{J}(\mathcal{S}_t) = & \max_{-v \leq H_t \leq A_1} \frac{1}{R} \left(\mathbb{E}^Q[\varphi(\mathcal{S}_{t+1}(H_t)) + d_{t+1}(H_t) + \tilde{J}(\mathcal{S}_{t+1}(H_t)) | \mathcal{S}_t = 1] \right) \\ = & \max_{-v \leq H_t \leq A_1} \frac{1}{R} G(H_t) \end{aligned} \quad (14)$$

where $G(H_t) = \mathbb{E}^Q[\varphi(\mathcal{S}_{t+1}(H_t)) + d_{t+1}(H_t) + \tilde{J}(\mathcal{S}_{t+1}(H_t)) | \mathcal{S}_t = 1]$. We analyze at each term of the function G individually.

The first term can be written as $\mathbb{E}^Q[\varphi(\mathcal{S}_{t+1}(H_t)) | \mathcal{S}_t = 1] = (\tau_D D + \tau_A v)\pi(H_t)$.

The second term can be written as

$$\begin{aligned} \mathbb{E}^Q[d_{t+1}(H_t) | \mathcal{S}_t = 1] = & \mathbb{E}^Q[(V_{t+1}(H_t) - R_D D - kD) \mathbb{1}_{[V_L, \infty)}(V_{t+1}(H_t)) | \mathcal{S}_t = 1] \\ = & \mathbb{E}^Q[V_{t+1}(H_t) \mathbb{1}_{[V_L, \infty)}(V_{t+1}(H_t)) | \mathcal{S}_t = 1] - (R_D D + kD)\pi(H_t). \end{aligned}$$

Since $\mathbb{E}^Q[V_{t+1}(H_t) \mathbb{1}_{[V_L, \infty)}(V_{t+1}(H_t)) | \mathcal{S}_t = 1] = qV_u(H_t) \mathbb{1}_{[V_L, \infty)}(V_u(H_t)) + (1 - q)V_d(H_t) \mathbb{1}_{[V_L, \infty)}(V_d(H_t))$

it follows from Lemma 1 and (13) that

$$\mathbb{E}^Q[d_{t+1}(H_t) | \mathcal{S}_t = 1] = qV_u(H_t) \mathbb{1}_{[V_L, \infty)}(V_u(H_t)) + (1 - q)V_d(H_t) \mathbb{1}_{[V_L, \infty)}(V_d(H_t)) - (R_D D + kD)\pi(H_t).$$

The last term of G can be written as $\mathbb{E}^Q[\tilde{J}(\mathcal{S}_{t+1}(H_t)) | \mathcal{S}_t = 1] = \tilde{J}(1)\pi(H_t)$ since $\tilde{J}(0) = 0$ in the case where there is a bank run between time t and time $t + 1$. Thus,

G can be written as

$$G(H_t) = (\tau_D D + \tau_A v) \pi(H_t) + q V_u(H_t) \mathbb{1}_{[V_L, \infty)}(V_u(H_t)) + (1-q) V_d(H_t) \mathbb{1}_{[V_L, \infty)}(V_d(H_t)) \\ - (R_D D + kD) \pi(H_t) + \tilde{J}(1) \pi(H_t)$$

and the dynamic programming equation (14) becomes

$$\begin{aligned} \tilde{J}(1) &= \max_{-v \leq H_t \leq A_1} \frac{1}{R} \{ q V_u(H_t) \mathbb{1}_{[V_L, \infty)}(V_u(H_t)) + (1-q) V_d(H_t) \mathbb{1}_{[V_L, \infty)}(V_d(H_t)) \\ &\quad + \pi(H_t) (\tau_D D + \tau_A v - R_D D - kD + \tilde{J}(1)) \} \\ &= \max_{-v \leq H_t \leq A_1} \frac{1}{R} \{ q (V_u(H_t) - R_D D - kD) \mathbb{1}_{[V_L, \infty)}(V_u(H_t)) \\ &\quad + (1-q) (V_d(H_t) - R_D D - kD) \mathbb{1}_{[V_L, \infty)}(V_d(H_t)) \\ &\quad + \pi(H_t) (\tau_D D + \tau_A v + \tilde{J}(1)) \} \\ &= \max_{-v \leq H \leq A_1} \frac{1}{R} \{ q (V_u(H) - R_D D - kD) \mathbb{1}_{[V_L, \infty)}(V_u(H)) \\ &\quad + (1-q) (V_d(H) - R_D D - kD) \mathbb{1}_{[V_L, \infty)}(V_d(H)) \\ &\quad + \pi(H) (\tau_D D + \tau_A v + J(1)) \}. \end{aligned}$$

where the last equality follows from the fact that (21) is homogeneous with respect to time. \square

Proof of Lemma 2. In Lemma 3 we found the following form of the dynamic programming equation:

$$\begin{aligned} J(1) &= \max_{-v \leq H \leq A_1} \frac{1}{R} \{ q (V_u(H) - R_D D - kD) \mathbb{1}_{[V_L, \infty)}(V_u(H)) \\ &\quad + (1-q) (V_d(H) - R_D D - kD) \mathbb{1}_{[V_L, \infty)}(V_d(H)) \\ &\quad + \pi(H) (\tau_D D + \tau_A v + J(1)) \}. \end{aligned}$$

Since $J(1)$ is a real number, we can write this equation equivalently as

$$y = \max_{-v \leq H \leq A_1} g(H, y) \tag{22}$$

where

$$\begin{aligned}
g(H, y) = & \frac{1}{R} \{ q(V_u(H) - R_D D - kD) \mathbb{1}_{[V_L, \infty)}(V_u(H)) \\
& + (1 - q)(V_d(H) - R_D D - kD) \mathbb{1}_{[V_L, \infty)}(V_d(H)) \\
& + \pi(H)(\tau_D D + \tau_A v + y) \}
\end{aligned} \tag{23}$$

Hence, we have to find a fix-point such that (22) holds. For convenient notation we define

$$K = D(R_D + k - \tau).$$

Then, $g(H, y)$ can be written as

$$g(H, y) = \frac{1}{R} \{ q(V_u(H) - K + y) \mathbb{1}_{[V_L, \infty)}(V_u(H)) + (1 - q)(V_d(H) - K + y) \mathbb{1}_{[V_L, \infty)}(V_d(H)) \}. \tag{24}$$

In the case where $\tilde{K}^d < \tilde{K}^u$, hedging positions $-v \leq H < \tilde{K}^u$ result in asset value below V_L in both state \mathcal{U} and \mathcal{D} (see Lemma 1). For hedging positions $H \geq \tilde{K}^u$, only the first term in (24) involving $V_u(H)$ remains. This term $V_u(H)$ is a increasing function in H , since $V_u(H) = vu + H(u - R)$ according to (7). Hence, the only candidate and also the optimal hedging position is A_1 .

The interesting case is when $\tilde{K}^d \geq \tilde{K}^u$. Choose any $y \in \mathbb{R}_+$ fixed. Now, hedging positions $[-v, \tilde{K}^d]$ lead to a positive payoff to shareholders in both state \mathcal{U} and \mathcal{D} (see Lemma 1). Thus,

$$g(H, y) = \frac{1}{R} \{ q(V_u(H) - K + y) + (1 - q)(V_d(H) - K + y) \} = v + \frac{-K + y}{R}$$

since $\frac{1}{R} \{ qV_u(H) + (1 - q)V_d(H) \} = v$. From Lemma 1 follows that the payoff to shareholders is zero in state \mathcal{D} for $H \geq \tilde{K}^d$. For such positions, only the first term in (24) remains. Hence, $g(\cdot, y)$ is of the following form:

$$g(H, y) = \begin{cases} v + \frac{-K + y}{R}, & H \in [-v, \tilde{K}^d], \\ \frac{q}{R}(vu + H(u - R) - K + y), & H \geq \tilde{K}^d \end{cases} \tag{25}$$

or, equivalently,

$$g(H, y) = \left(v + \frac{-K + y}{R} \right) \mathbb{1}_{[-v, \tilde{K}^d]}(H) + \frac{q}{R} (vu + H(u - R) - K + y) \mathbb{1}_{[\tilde{K}^d, \infty)}(H)$$

In other words, $g(\cdot, y)$ is constant on $[-v, \tilde{K}^d]$ and increasing for $H \geq \tilde{K}^d$. Using (25), we can therefore define, for any fixed $y \in \mathbb{R}_+$

$$g_1(y) := \max_{-v \leq H \leq \tilde{K}^d} g(H, y) = \left(v + \frac{-K + y}{R} \right). \quad (26)$$

and

$$g_2(y) := \max_{\tilde{K}^d \leq H \leq A_1} g(H, y) = \frac{q}{R} (vu + A_1(u - R) - K + y). \quad (27)$$

Since $g(\cdot, y)$ is increasing for $H \geq \tilde{K}^d$, the maximum in $g_2(y)$ will be attained in A_1 , $g_2(y) = g(A_1, y)$. The problem (22) can now be written as

$$y = \max_{-v \leq H \leq A_1} g(H, y) = \max\{g_1(y), g_2(y)\}.$$

We next look at the two problems $y = g_i(y), i = 1, 2$, individually. In the proof of Proposition 1, the problem for the maximum is discussed. Using definition (26) of g_1 and solving the equation $\bar{y}_1 = g_1(\bar{y}_1)$ for \bar{y}_1 yields

$$\bar{y}_1 = \frac{1}{R-1} (Rv - K); \quad (28)$$

using definition (27) of g_2 and solving the equation $\bar{y}_2 = g_2(\bar{y}_2)$ for \bar{y}_2 yields

$$\bar{y}_2 = \frac{q}{R-q} (vu + A_1(u - R) - K). \quad (29)$$

\bar{y}_1 is positive if and only if $v > \frac{K}{R}$ which yields condition (17); \bar{y}_2 is positive if and only if $vu + A_1(u - R) > K$ which yields condition (15). $\bar{y}_i, i = 1, 2$, are candidates for $J(1)$ in (21). $\tilde{J}(1)$ in (16) is obtained by substituting in \bar{y}_2 the definition of K , adding the rent of the first period τD , since $q = \pi(A_1)$. Analogously, $\tilde{J}(1)$ in (18) is obtained by substituting in \bar{y}_1 the definition of K , adding the rent of the first period τD , since $1 = \pi(H)$ for $H \in [\tilde{K}^u, \tilde{K}^d]$.

□

Proof of Proposition 1. Case 1 is obvious from the remarks given in the proof of Lemma 2 on the case $\tilde{K}^d < \tilde{K}^u$, since $vR < V_L$ is, according to Lemma 1, equivalent to $\tilde{K}^d < \tilde{K}^u$.

In the proof of Lemma 2, we found the candidates \bar{y}_i , $i = 1, 2$ for a fixed point of (22). What remains to be shown is that also $\bar{y}_i = \max\{g_1(\bar{y}_i), g_2(\bar{y}_i)\} = \max_{-v \leq H \leq A_1} g(H, \bar{y}_i)$ for $i = 1$ or $i = 2$ holds. Using the notation of the proof of Lemma 2, we find the following for the case where $vR \geq V_L$, i.e., $\tilde{K}^d \geq \tilde{K}^u$:

1. If $g_2(\bar{y}_2) > g_1(\bar{y}_2)$ and $g_2(\bar{y}_1) > g_1(\bar{y}_1)$, then the maximum is attained in A_1 . Indeed, the equality $\bar{y}_2 = \max\{g_1(\bar{y}_2), g_2(\bar{y}_2)\} = g_2(\bar{y}_2)$ holds, but $\max\{g_1(\bar{y}_1), g_2(\bar{y}_1)\} = g_2(\bar{y}_1) \neq \bar{y}_1$, hence, the maximum is attained in A_1 .
2. If $g_1(\bar{y}_1) > g_2(\bar{y}_1)$ and $g_1(\bar{y}_2) > g_2(\bar{y}_2)$, then the maximum is attained on the set $[-v, \tilde{K}^d]$. Indeed, the equality $\bar{y}_1 = \max\{g_1(\bar{y}_1), g_2(\bar{y}_1)\} = g_1(\bar{y}_1)$ holds, but $\max\{g_1(\bar{y}_2), g_2(\bar{y}_2)\} = g_1(\bar{y}_2) \neq \bar{y}_2$, hence, the maximum is attained on the set $[-v, \tilde{K}^d]$.
3. If $g_1(\bar{y}_1) = \bar{y}_1 = g_2(\bar{y}_1)$ and $g_1(\bar{y}_2) = g_2(\bar{y}_2) = \bar{y}_2$, then the maximum is attained both on the set $[-v, \tilde{K}^d]$ and in A_1 . Indeed, the equality $\bar{y}_1 = \max\{g_1(\bar{y}_1), g_2(\bar{y}_1)\} = g_1(\bar{y}_1) = \bar{y}_2 = \max\{g_1(\bar{y}_2), g_2(\bar{y}_2)\} = g_1(\bar{y}_2)$ holds and the maximum is attained both on the set $[-v, \tilde{K}^d]$ and in A_1 .

The following Lemma gives equivalent conditions for these cases to happen.

- Lemma 4.**
1. If $\bar{y}_1 < \bar{y}_2$, then $g_2(\bar{y}_2) > g_1(\bar{y}_2)$ and $g_2(\bar{y}_1) > g_1(\bar{y}_1)$, that is, the maximum is attained in A_1 .
 2. If $\bar{y}_1 > \bar{y}_2$, then $g_1(\bar{y}_1) > g_2(\bar{y}_1)$ and $g_1(\bar{y}_2) > g_2(\bar{y}_2)$, i.e., the maximum is attained on the set $[-v, \tilde{K}^d]$.
 3. If $\bar{y}_2 = \bar{y}_1$, then $g_2(\bar{y}_2) = g_1(\bar{y}_2)$ and $g_2(\bar{y}_1) = g_1(\bar{y}_1)$, i.e., the maximum is attained both on the set $[-v, \tilde{K}^d]$ and in A_1 .

Proof. Consider the case where $\bar{y}_1 < \bar{y}_2$. By definition, $g_1(\bar{y}_1) = \bar{y}_1$. The slope of g_2

is $q/R < 1$, by definition (27). Then,

$$\begin{aligned} g_2(\bar{y}_1) &= g_2(\bar{y}_2) + (\bar{y}_1 - \bar{y}_2)(q/R) = \bar{y}_2(1 - q/R) + (q/R)\bar{y}_1 \\ &> \bar{y}_1(1 - q/R) + (q/R)\bar{y}_1 = \bar{y}_1 = g_1(\bar{y}_1). \end{aligned}$$

Similarly, the slope of g_1 is, by definition (26), $1/R < 1$. Thus

$$\begin{aligned} g_1(\bar{y}_2) &= g_1(\bar{y}_1) + (1/R)(\bar{y}_2 - \bar{y}_1) = \bar{y}_1 + (1/R)(\bar{y}_2 - \bar{y}_1) = \bar{y}_1(1 - 1/R) + (1/R)\bar{y}_2 \\ &< \bar{y}_2(1 - 1/R) + (1/R)\bar{y}_2 = \bar{y}_2 = g_2(\bar{y}_2). \end{aligned}$$

The second case follows analogously. Suppose $\bar{y}_1 > \bar{y}_2$. By definition, $g_1(\bar{y}_1) = \bar{y}_1$. The slope of g_2 is, by definition (27), $q/R < 1$. Then,

$$\begin{aligned} g_2(\bar{y}_1) &= g_2(\bar{y}_2) + (\bar{y}_1 - \bar{y}_2)(q/R) = \bar{y}_2(1 - q/R) + (q/R)\bar{y}_1 \\ &< \bar{y}_1(1 - q/R) + (q/R)\bar{y}_1 = \bar{y}_1 = g_1(\bar{y}_1). \end{aligned}$$

Similarly, the slope of g_1 is, by definition (26), $1/R < 1$. Thus

$$\begin{aligned} g_2(\bar{y}_2) &= \bar{y}_2 = \bar{y}_1 - (\bar{y}_1 - \bar{y}_2) < \bar{y}_1 - (1/R)(\bar{y}_1 - \bar{y}_2) \\ &= g_1(\bar{y}_1) + (1/R)(\bar{y}_2 - \bar{y}_1) = g_1(\bar{y}_2). \end{aligned}$$

In the case where $\bar{y}_2 = \bar{y}_1$, we have that $g_1(\bar{y}_1) = \bar{y}_1 = \bar{y}_2 = g_2(\bar{y}_2)$. This completes the proof of Lemma 4. \square

What is left to prove for Proposition 1 is that the condition $\bar{y}_1 < \bar{y}_2$ can be

equivalently formulated as $\tau < \tau_*$. Using definitions (28) and (29), we find that

$$\begin{aligned}
& \bar{y}_1 < \bar{y}_2 \\
& \Leftrightarrow \frac{R}{R-1} \left(v - \frac{K}{R} \right) < \frac{q}{R-q} (vu + A_1(u-R) - K) \\
& \Leftrightarrow \frac{R}{R-1} \left(v - \frac{D}{R} (R_D + k - \tau) \right) < \frac{q}{R-q} (vu + A_1(u-R) - D(R_D + k - \tau)) \\
& \Leftrightarrow \tau D \left(\frac{1}{R-1} - \frac{q}{R-q} \right) < \frac{q}{R-q} (vu + A_1(u-R)) - \frac{Rv}{R-1} + D(R_D + k) \left(\frac{1}{R-1} - \frac{q}{R-q} \right) \\
& \Leftrightarrow \tau < \frac{\frac{q}{R-q} (vu + A_1(u-R)) - \frac{Rv}{R-1}}{D \left(\frac{1}{R-1} - \frac{q}{R-q} \right)} + (R_D + k)
\end{aligned}$$

where the right hand side of the last inequality can be simplified to yield τ_* . The last equivalence holds because $\frac{1}{R-1} - \frac{q}{R-q} = \frac{R(1-q)}{(R-1)(R-q)} > 0$. \square

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