The Impact of Heterogeneity of Beliefs on Trading Volume*

Wolfgang Bauer

First draft: September 10th, 2003
This draft: June 1st, 2004

Abstract

This paper analyzes the effects that uncertainty about economic fundamentals has on aggregate trading volume. First, the trading volume of an investor facing a standard consumption portfolio choice problem is derived. It is found that if the parameters describing the investment opportunity set follow diffusion processes, also the trading volume follows under general conditions a diffusion process. We then analyze a pure exchange economy of the type studied by Detemple and Murthy (1994), Basak (2000) where the growth rate of aggregate consumption is unobservable and investors have differing beliefs on its dynamics. It is found that disagreement on the mean growth rate increase the absolute size of expected trading volume. This effect of the disagreement is stronger when investors also have different beliefs on the variance of the growth rate.

*We wish to thank the National Centre of Competence in Research 'Financial Valuation and Risk Management' (NCCR FINRISK) for financial support; NCCR FINRISK is a research programme by the Swiss National Science Foundation. We gratefully acknowledge helpful comments from Rajna Gibson as well as advice from Jérôme Detemple on an early version of the paper.

1 Introduction

Fundamentals of the economy are of central concern to financial markets. In general, views differ widely on the prospects of economies, notwithstanding all the efforts spent on the analysis of the fundamentals. A natural question that arises then is how the disagreement among agents in an economy affects their trading on financial markets. In this paper, we will concentrate on the agents' trading volume and the impact that differences in expectations on the future economic development has on trading volume.

An issue that has so far not been analyzed is how the disagreement among investors affects the trading volume in the economy. A large body of the literature has analyzed the question how heterogeneity of beliefs among agents in an economy affect equilibrium prices, returns and interest rates (see for example Detemple and Murthy (1994), Zapatero (1998), Basak (2000) and the literature cited therein). The focus of this literature is on the expected growth rate of aggregate consumption as an unobservable quantity. Investors all observe realized growth rates, thus having symmetric, but incomplete information. Because they do not know the true initial value of the expected growth rate but perceive it to be random according to different probability distributions, they cannot disentangle the expected value and the stochastic component of growth rates. In this situation, the no-trade result of Milgrom and Stokey (1982) does not apply which says that agents with different beliefs will not trade upon receiving new public information if and only if the relative conditional likelihoods of any two states is equal across agents (see also Grundy and McNichols (1989)).

While the focus of this paper is on the effects that disagreement on the economic fundamentals has on aggregate trading volume, a large literature has analyzed the effect that private information on individual stocks has on the trading volume in that particular stocks, that is, informational asymmetry is a key feature of these models. Informational asymmetry is at the core of an important part of the market microstructure where market makers face investors some of which have better information on the fundamental value of the stock and need to set their quoted prices and quantities (for an overview on the market micro-structure literature, see for example O'Hara (1995), Biais et al. (2002)). Among the important driving forces of this research was the empirical finding that absolute price changes are positively correlated with trading volume (Ying, 1966, Karpoff, 1987, Jain and Joh, 1988, Gallant et al., 1992, Gerety and Mulherin, 1992). A large number of models assume that the volume-price relationship is due to unequal information among market participants. The models of sequential arrival of information (Copeland, 1976, 1977, Morse, 1980, Jennings et al., 1981, Jennings and Barry, 1983) assume that information reaches traders at a different time. Uninformed traders do not learn the information from the trading of the informed traders. In other models, uninformed traders try to infer from the observed trading volume the information they do not have access to (Kyle, 1985, Admati, 1988, Foster and Viswanathan, 1990, 1993).

Another direction of the literature on differences in beliefs incorporates besides better informed market participants also liquidity traders whose trading activity hinders investors to infer the private information (Harris and Raviv, 1993, Shalen, 1993, Kandel and Pearson, 1995, Biais and Bossaerts, 1998). This generally increases volatility. Shalen (1993) finds that this additional noise increases expected volume. In some models, such as Blume et al. (1994), agents learn from observing prices and volume the precision of the information of other agents.

A key element in this literature is inequality of information available to market participants. This is an important aspect on the level of individual stocks and firms. Yet, it is unlikely that some investors have insider information on the state of the general economy. Therefore, the results found in the asymmetric information literature do not directly apply to the case of uncertainty about economic fundamentals. Thus, this paper focuses on the situation where no market participant has an informational advantage. Rather, all agents are uncertain about the economic fundamentals. Therefore, information is symmetrically distributed, but all agents have incomplete information and there is no possibility that information sharing could reveal the true state. Agents can observe others' beliefs but cannot assess whether these are correct or not.

Most closely related to this paper is the work of Buraschi and Jiltsov (2002, 2003) who have analyzed the trading volume of options in a similar economic setting with differences in beliefs. They have found that the option trading volume is sensitive with respect to changes in the differences of beliefs.

This paper builds on the well-understood workhorse model in financial economics of an investor's consumption and portfolio choice (see e.g. Merton (1969, 1971), Cox and Huang (1989, 1991), Karatzas et al. (1986)). Ocone and Karatzas (1991) have derived the structure of the optimal portfolio. In a first contribution, the trading volume of an individual investor is derived. The expression found for the money amount traded in the different stocks holds for an investor with power utility and a general process of the interest rate and the market price of risk. We show that in the general case of an economy where state variables follow diffusion processes, volume is given by a diffusion process as well. It is found that trading has different motivations, coming from the demand for the mean-variance efficient portfolio and hedging portfolios in the sense of Merton (1973). We perform a numerical analysis of the expected trading volume and its volatility and find that the largest part of the expected volume stems from rebalancing the mean-variance portfolio and that therefore time horizon has little effect on expected volume. Risk aversion reduces both the expected size of trading volume related to the mean-variance portfolio as well as its volatility. Yet, trading volume related to the hedging portfolio is more volatile for investors with a higher risk aversion, because they adjust their holdings of the hedging portfolios. Both the expected volume from the hedging portfolio as well as its volatility change sign at a risk aversion of 1.

In a next step, the model is extended to an equilibrium model which is a continuous time version of the Radner (1972) pure-exchange model, but with heterogeneous beliefs, similar to Detemple and Murthy (1994), Zapatero (1998), Basak (2000). It is a pure exchange economy where agents have incomplete information on the expected growth rate and have heterogeneous beliefs about the unobservable growth rate. We analyze the effect that varying degrees of disagreement have on the aggregate trading volume and find that disagreement on the mean increases the absolute size of the expected trading volume as well as its volatility and that additional disagreement on the variance makes this effect more pronounced. A longer investment horizon as well as lower risk aversion tends to induce higher expected trading volume. For investors with low relative risk aversion, volatility of trading volume is the highest for a short investment time horizon. To reproduce from the model actual trading volume, investors disagree both on the mean and variance. In general, trading due to hedging reasons contributes a large part to the expected trading volume, while the hedging and mean-variance component are equally important for the volatility of trading volume.

The remaining text is organized as follows. In section 2, the financial market and the investor's portfolio problem is summarized, the general expressions for a utility maximizing investor's trading volume is derived and its properties are explored in a numerical study. In section 3, an equilibrium model is set up where investors have incomplete but symmetric information about the growth rate of aggregate consumption and the effects of heterogeneity of beliefs on trading volume are analyzed. Section section 4 concludes.

2 Trading volume of an individual investor

2.1 The financial market

The subject of this paper is an economy with a capital market satisfying the usual assumptions of a perfect market. In particular (see e.g. Merton (1973)), there are no transactions costs or taxes, assets are perfectly divisible, the capital market is always in equilibrium, lending and borrowing is possible at the same rate, short-selling is allowed and trading takes place continually in time.

In this section, the trading volume of a utility maximizing investor is derived, based on the standard models of consumption and portfolio choice (see Merton (1969, 1971), Cox and Huang (1989, 1991), Karatzas et al. (1986)). Throughout the paper, we consider a complete financial market consisting of a bank account and d stocks. The value of the bank account has the dynamics

$$dB(t) = B(t)r(t)dt,\ B(0) = 1$$

The gains on stock $j, j = 1, 2, \dots, d,$ that is the capital gain plus the dividend yield, have the dynamics

$$dG_j(t) \equiv \frac{dS_j(t)}{S_j(t)} + \frac{\delta_j(t)dt}{S_j(t)} = \mu_j(t)dt + \sigma_j(t)dW(t),\ S_j(0) = S_{j,0}, \qquad (1)$$

where $\delta_j(t)$ denotes the dividend rate, $S_{j,0}$ is a given constant, $\mu_j(t)$ the expected rate of return, and $\sigma_j(t) = (\sigma_{1j}(t), \dots, \sigma_{dj}(t))$ the diffusion term. The d sources of uncertainty are modelled by the $d-$dimensional Brownian motion $W(t) = (W^1(t), \dots, W^d(t))'$, $0 \leq t \leq T$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The filtration generated by the Brownian motion is denoted by $\{\mathcal{F}(t)\}_{0 \leq t \leq T}$ and satisfies the usual assumptions. The investment opportunity set is stochastic, that is, the interest rate as well as the expected return and the volatility of the stocks evolve stochastically. Uncertainty about these quantities is represented by p state variables[1] $Y_j(t)$, $j = 1, 2, \dots, p$, which follow the vector diffusion process

$$dY(t) = \mu^Y(t, Y(t))dt + \sigma^Y(t, Y(t))dW(t),\ Y(0) = Y_0, \qquad (2)$$

where $\mu^Y : [0, T] \times \mathbb{R}^p \to \mathbb{R}^p$ and $\sigma^Y : [0, T] \times \mathbb{R}^p \to \mathbb{R}^{p \times d}$ and we assume that $\mu^Y \in C^{1,2}([0, T] \times \mathbb{R}^p)$ and $\sigma^Y \in C^{1,2}([0, T] \times \mathbb{R}^p)$ componentwise. Both μ^Y and σ^Y satisfy growth conditions. The initial value $Y_0 \in \mathbb{R}^p$ is given. We assume that the interest rate earned on the bond is a function of the state variables,

[1] On page 10, the relationship between the number of state variables p, the number of Brownian motions d and the number of traded assets d is discussed.

$r(t) = r(t, Y(t))$, where we assume that $r \in C^{1,2}([0,T] \times \mathbb{R}^p)$. The expected rate of appreciation on the stocks is as well a function of the state variables, $\mu_j(t) = \mu_j(t, Y(t))$, as are also the dividend rate by $\delta_j(t) = \delta_j(t, Y(t))$ and the volatility coefficients $\sigma_j(t) = (\sigma_{1j}(t), \ldots, \sigma_{dj}(t))$ where $\sigma_{ij}(t) = \sigma_{ij}(t, Y(t))$, $i = 1, 2, \ldots, d$. The $(d \times d)$ matrix $\Sigma(t)$ has as $j-$th row the vector $\sigma_j(t)$ and is assumed to be invertible for almost all $(\omega, t) \in \Omega \times [0, T]$. The vector $\mu(t) = (\mu_1(t), \ldots, \mu_d(t))'$ consists of the expected rates of appreciation of the individual stocks. The market price of risk is given by

$$\theta(t) = \theta(t, Y(t)) \equiv \Sigma(t)^{-1}(\mu(t) - r(t)\mathbf{1})$$

where $\mathbf{1} \in \mathbb{R}^d$ denotes a vector of ones and $\theta \in C^{1,2}([0,T] \times \mathbb{R}^p)$ is assumed. Assuming that $\{\theta(t)\}_{0 \leq t < \infty}$ satisfies the Novikov condition $\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \theta(t)'\theta(t)dt\right)\right] < \infty$, we define by

$$\chi(t) \equiv \exp\left(-\int_0^t \theta(u)'dW(u) - \frac{1}{2}\int_0^t \theta(u)'\theta(u)du\right) \tag{3}$$

the density of the equivalent martingale measure, $\frac{dQ}{d\mathbb{P}}|_{\mathcal{F}_T} = \xi(T)$. The state price density that that yields the arbitrage free price of cash flows is then defined as

$$\xi(t) \equiv \chi(t)\left(-\int_0^t r(u)du\right) = \exp\left(-\int_0^t \theta(u)'dW(u) - \frac{1}{2}\int_0^t \theta(u)'\theta(u)du - \int_0^t r(u)du\right). \tag{4}$$

2.2 The investor's optimization problem

We will look at two different problems: In the first situation, that we call the investment problem, the investor has utility $B(X(T))$ only from wealth $X(T)$ at time T, the end of his time horizon, and seeks to invest his present wealth x such as to maximize his expected utility. His investment strategy is given by a adapted process $\pi = \{\pi(t)\}_{0 \leq t \leq T}$. In the second problem, we call it the consumption problem, the investor has time-additive utility from intermediate consumption, $u(c(t), t)$, $0 \leq t \leq T$, before the end of this time horizon. In this case he looks for the optimal investment strategy $\pi = \{\pi(t)\}_{0 \leq t \leq T}$ as well as the optimal consumption policy $c = \{c(t)\}_{0 \leq t \leq T}$. When allocating amounts of size $\pi(t) = (\pi_1(t), \ldots, \pi_d(t))'$ to the d stocks, consuming at a rate $c(t) \geq 0$ and investing the remaining into the bank account, his wealth $X(t)$ has the dynamics

$$dX(t) = (X(t)r(t) - c(t))dt + \pi(t)'[(\mu(t) - r(t)\mathbf{1})dt + \Sigma(t)dW(t)].$$

Problem 1 (Investment problem). The investor's goal is to maximize his expected utility from terminal wealth such that his wealth is nonnegative at all times, $X(t) \geq 0$, $0 \leq t \leq T$. The equivalent static budget constraint to this solvency constraint is given by (Cox and Huang, 1989, 1991)

$$\max_{\pi} \mathbb{E}\left[B(X(T))\right]$$
$$s.t.\, \mathbb{E}\left[\xi(T)X(T)\right] \leq x.$$

Problem 2 (Consumption problem). The investor's goal is to maximize his total expected utility from consumption such that his wealth is nonnegative at all times, $X(t) \geq 0$, $0 \leq t \leq T$. The static problem writes as

$$\max_{\pi} \mathbb{E}\left[\int_0^T u(c(t),t)dt\right]$$
$$s.t.\, \mathbb{E}\left[\int_0^T \xi(t)c(t)dt\right] \leq x.$$

2.3 The structure of the optimal stock portfolio

2.3.1 General utility function

The optimal stock portfolio in Problem 1 and 2 has the following structure (Karatzas and Shreve, 1998):

$$\pi(t) = X(t)(\Sigma(t)')^{-1}\theta(t) + \xi(t)^{-1}(\Sigma(t)')^{-1}\phi(t).$$

The process $\phi(t)$ is the integrand in the stochastic integral representation $M(t) = \mathbb{E}\left[M(T)\right] + \int_0^t \phi(t)'dW(t)$ of the following martingales:

- In the investment Problem 1,

$$M(t) = \mathbb{E}\left[\xi(T)X_i(T)|\mathcal{F}(t)\right],$$

  where the optimal wealth is

$$X_i(T) \equiv I_B(y\xi(T)) \tag{5}$$

  and $I_B(\cdot)$ denotes the inverse function of the marginal utility of wealth $\partial B(X)/\partial X$.

- In the consumption Problem 2,

$$M(t) = \mathbb{E}\left[\int_0^T \xi(u)\hat{c}(u)du|\mathcal{F}(t)\right],$$

where the optimal consumption is given by

$$\hat{c}(t) \equiv I(y\xi(t), t) \tag{6}$$

where $I(\cdot, t)$ is the inverse function of the marginal utility of consumption $\partial u(c, \cdot)/\partial c$, y is the Lagrange multiplier that solves the budget constraint

$$x = \mathbb{E}\left[\int_0^T \xi(t) I(y\xi(t), t) dt\right]. \tag{7}$$

Ocone and Karatzas (1991) have derived the structure of $\phi(t)$ for Problems 1 and 2 and found the following form of the optimal stock portfolio :

$$\pi(t) = \pi^{(mv)}(t) w^{(mv)}(t) + (\Sigma(t)')^{-1} \bar{w}^{(h)}(t). \tag{8}$$

where

$$\pi^{(mv)}(t) \equiv (\Sigma(t)')^{-1}\theta, \tag{9}$$

$$w^{(mv)}(t) \equiv \begin{cases} \mathbb{E}\left[\frac{\xi(T)}{\xi(t)} \frac{X_i(T)}{R_B(X_i(T))} | \mathcal{F}(t)\right] \text{ for the investment problem,} \\ \mathbb{E}\left[\int_t^T \frac{\xi(v)}{\xi(t)} \frac{1}{R_c(\hat{c}(v), v)} \hat{c}(v) dv | \mathcal{F}(t)\right] \text{ for the consumption problem,} \end{cases} \tag{10}$$

$$\bar{w}^{(h)}(t) \equiv \begin{cases} -\mathbb{E}\left[\frac{\xi(T)}{\xi(t)} X_i(T)\left(1 - \frac{1}{R_B(X_i(T))}\right) H_t(T) | \mathcal{F}(t)\right] \text{ for the investment problem,} \\ -\mathbb{E}\left[\int_t^T \frac{\xi(v)}{\xi(t)} \hat{c}(v)\left(1 - \frac{1}{R_c(\hat{c}(v), v)}\right) H_t(v) dv | \mathcal{F}(t)\right] \text{ for the consumption problem,} \end{cases} \tag{11}$$

$$R_U(c) \equiv -\frac{cU''(c)}{U'(c)},\ c \geq 0, \tag{12}$$

$$R_B(X)) \equiv -\frac{XB''(X)}{B'(X)},\ X \geq 0, \tag{13}$$

$$H_t(v)' \equiv \int_t^v \mathcal{D}_t\left(r(u)\right) du + \int_t^v (dW(u) + \theta(u)du)' \mathcal{D}_t\left(\theta(u)\right), v \geq t. \tag{14}$$

The investor's total demand for stocks, $\pi(t)$ as defined in (8), consists of two components. The first results from investing an amount of $w^{(mv)}(t)$ dollars into the instantaneously mean-variance efficient portfolio

$$\pi^{(mv)}(t) = (\Sigma(t)')^{-1}\theta(t) = (\Sigma(t)\Sigma(t)')^{-1}(\mu(t) - r(t)\mathbf{1}). \tag{15}$$

We call $w^{(mv)}(t)$, as defined in (10), the demand for the mean-variance efficient portfolio. It depends on the future wealth (consumption) and relative risk aversion of wealth $R_B(X)$ (relative risk aversion of consumption $R_U(c)$ respectively).

Since $w^{(mv)}(t)$ is a conditional expectation, it is stochastic. The second component of the stock demand, $(\Sigma(t)')^{-1}\bar{w}^{(h)}(t)$, involves the term $H_t(v)'$, $v \geq t$, which depends on the Malliavin derivatives of the interest rate and the market price of risk, $\mathcal{D}_t\left(r(u)\right), u \geq t,$ and $\mathcal{D}_t\left(\theta(u)\right), u \geq t.$ By writing the optimal portfolio (8) in an equivalent form, it becomes clear that this second component, it will be called the hedging demand, is held to hedge against shifts in the interest rate and the market price of risk.[2]

Proposition 1. The optimal stock portfolio has the structure

$$\pi(t) = \pi^{(mv)}(t)w^{(mv)}(t) + \pi^{(h)}(t)w^{(h)}(t) \tag{16}$$

where [3]

$$\pi^{(h)}(t) \equiv (\Sigma(t)\Sigma(t)')^{-1}\Sigma(t)\sigma^Y(t)', \tag{17}$$

$$w^{(h)}(t) \equiv \begin{cases} -\mathbb{E}\left[\frac{\xi(T)}{\xi(t)}X_i(T)\left(1-\frac{1}{R_B(X_i(T))}\right)h_t(T)|\mathcal{F}(t)\right] \text{ for the investment problem,} \\ -\mathbb{E}\left[\int_t^T \frac{\xi(v)}{\xi(t)}\hat{c}(v)\left(1-\frac{1}{R_c(\hat{c}(v),v)}\right)h_t(v)dv|\mathcal{F}(t)\right] \text{ for the consumption problem,} \end{cases} \tag{18}$$

$$h_t(v)' \equiv h_t^r(v)' + h_t^\theta(v)',\ v \geq t \tag{19}$$

$$h_t^r(v)' \equiv \int_t^v \partial_2 r(u)\Phi_t(u)du,\ v \geq t \tag{20}$$

$$h_t^\theta(v)' \equiv \int_t^v (dW(u)+\theta(u)du)'\partial_2\theta(u)\Phi_t(u), v \geq t, \tag{21}$$

and $\Phi_t(u)$, $t \leq u \leq T$ is the $(p \times p)$ matrix solving

$$d\Phi_t(u) = \partial_2\mu^Y(u)\Phi_t(u)du + \sum_{j=1}^d \partial_2\sigma_{\cdot j}^Y(u)\Phi_t(u)dW_j(u), \tag{22}$$

$$\Phi_t(t) = I. \tag{23}$$

Proof. See the Appendix. □

[2] I thank Jérôme Detemple for pointing out to me the decomposition of $H_t(v)$.

[3] Throughout the paper, we use the following notation: For a function $f : [0,T]\times\mathbb{R}^n \to \mathbb{R}^m$, we define the $(m \times n)$ Jacobian $\partial_2 f$ by its (i,j) element

$$J_{ij} = \frac{\partial f_i(t,y_1,\ldots,y_n)}{\partial y_j}, \quad 1 \leq i \leq m, 1 \leq j \leq n.$$

For a function $f : [0,T]\times\mathbb{R}^n \to \mathbb{R}^{m_1}\times\mathbb{R}^{m_2}$, we denote by $\partial_{2,2}f$ the $(m_1 \times n \times m_2)$ tensor with (i,j,k) element

$$J_{ijk} = \frac{\partial f_i^2(t,y_1,\ldots,y_n)}{\partial y_j \partial y_k}, \quad 1 \leq i \leq m_1, 1 \leq j \leq n, 0 \leq k \leq m_2.$$

According to this Proposition, the investor's stock demand $\pi(t)$ in (16) has the following structure for both the investment and the consumption problem: A first component, $\pi^{(mv)}(t)w^{(mv)}(t)$, comes from the investment into the mean-variance efficient portfolio, and a second component, $\pi^{(h)}(t)w^{(h)}(t)$, is due to the investment into a number of hedging portfolios.

The structure of these hedging portfolios is described by the $(d \times p)$ matrix $\pi^{(h)}(t)$ where we denote by $\pi_k^{(h)}(t)$ its k-th column. Each column $\pi_k^{(h)}(t)$ is the portfolio composed of the d risky assets which has maximal correlation with the k-th state variable $Y_k(t)$,

$$\pi_k^{(h)}(t) = (\Sigma(t)\Sigma(t)')^{-1}\Sigma(t)\sigma_k^Y(t)',$$

where $\sigma_k^Y(t)$ denotes the k-th row of $\sigma^Y(t)$. The structure of these portfolios is independent of the investor's preferences, wealth or time horizon. It is completely determined by the market parameters $\Sigma(t)$ and $\sigma^Y(t)$.

The amount of money that the investor invests into each of the hedging portfolios is given by the p-dimensional vector $w^{(h)}(t)$, defined in (18). We call the amount that he invests into the k-th hedging portfolio, $w_k^{(h)}(t)$, $k = 1, 2, \ldots, p$, the investor's demand for the k-th hedging portfolio. It depends on the investor's optimal wealth (consumption), hence on his utility function as well as the investment time horizon T and the term $h_t(v)$, $v \geq t$. We call the total amount that he invests into the i-th stock due to hedging,

$$\pi_i^{(h)}(t)w^{(h)}(t) = \sum_{k=1}^{p} \pi_{i,k}^{(h)}(t)w_k^{(h)}(t)$$

his hedging demand for the i-th stock.

The total amount that the investor holds in stock i, $i = 1, 2, \ldots, d$, is given by

$$\pi_i(t) = \pi_i^{(mv)}(t)w^{(mv)}(t) + \sum_{k=1}^{p} \pi_{i,k}^{(h)}(t)w_k^{(h)}(t),$$

where $\pi_{i,k}^{(h)}(t)$ denotes the (i,k) element of $\pi^{(h)}(t)$. This is Merton's $p + 2$ fund separation result (Merton, 1973) which says that all investors hold a combination of the same $p+2$ funds: The riskless asset on one hand and $p+1$ funds composed of the risky assets on the other hand. One fund is the instantaneously mean-variance efficient portfolio and each of the other p funds is a hedging portfolio, each hedging against shifts in a state variable. While the composition of the funds is identical across all investor's, the amounts they invest into the funds differs among them according to their risk aversion and wealth.

In general, the number of state variables can differ from the number of risky assets, as e.g. in Breeden (1979). If there are more sources of uncertainty than states variables, $p > d$, there will be redundant hedging portfolios in the sense that the returns on some of the hedging portfolios can be exactly replicated by a combination of the other hedging portfolios (Merton, 1990, p. 506). In this case, an analysis such as in Merton (1973) where investors compose their portfolios from the hedging portfolios instead of the original assets, becomes impossible, since the covariance matrix between the returns on the hedging portfolios would not be regular any more. Yet, the financial market in this paper is always complete, consisting of one bond and d stocks, where the number of traded assets equals the number of sources of uncertainty plus one.

Where Merton's demands for the hedging portfolios depend on the value function, resulting from a dynamic programming approach, the demand $w^{(h)}(t)$ involves the term $h_t(v)$. It is related to

$$H_t(v)' \equiv \int_t^v \mathcal{D}_t\left(r(u)\right) du + \int_t^v \left(dW(u) + \theta(u)du\right)' \mathcal{D}_t\left(\theta(u)\right), v \geq t,$$

used in previous literature. This latter term $H_t(v)$ captures the impact that a shock at time t to the economy, as represented by the Brownian motions $W(t)$, has on the investment opportunity set at time the future time v, given by $r(v)$ and $\theta(v)$. To understand the role of the new term $h_t(v)$, we notice that due to equality (83) in the Appendix from the proof of Proposition 1,

$$H_t(v)' = h_t(v)' \sigma^Y(t, Y(t)).$$

Thus, $h_t(v)$ represents the sensitivity of the investment opportunity set, given by the interest rate and the market price of risk, at the future time $v \geq t$ with respect to shocks to the state variables $Y(t)$ at time t (see Detemple et al. (2003)), in contrast to $H_t(v)$ which represents this sensitivity with respect to the Brownian motions $W(t)$ at time t. To hedge the portfolio against shocks to the k-th state variable $Y_k(t)$, an investor needs to go short in the k-th hedging portfolio by an amount equal to this sensitivity, $-h_{k,t}(v)$, $k = 1, \ldots, p$, where $h_{k,t}(v)$ the denotes the k-th element of the $(p \times 1)$ vector $h_t(v)$. This quantity required to insure against shocks differs across the states of the world, $h_{k,t}(v)$ is a random variable. It is independent of the investor's preferences or wealth, it is determined purely by the intertemporal dynamics of the investment opportunity set.

The extent to which an investor demands insurance against shocks to the k-th state variable depends on his optimal wealth (consumption) and his attitude towards risk,

$$w_k^{(h)}(t) = \begin{cases} \mathbb{E}\left[\frac{\xi(T)}{\xi(t)} X_i(T) \left(1 - \frac{1}{R_B(X_i(T))}\right)(-h_{k,t}(T))|\mathcal{F}(t)\right] & \text{for the investment problem,} \\ \mathbb{E}\left[\int_t^T \frac{\xi(v)}{\xi(t)} \hat{c}(v) \left(1 - \frac{1}{R_c(\hat{c}(v),v)}\right)(-h_{k,t}(v))\, dv|\mathcal{F}(t)\right] & \text{for the consumption problem.} \end{cases} \tag{24}$$

The higher the optimal wealth (consumption) in any fixed state of the world, the higher is the demand for the insurance. In the present general setting, relative risk aversion $R(\cdot)$ varies with the level of wealth (consumption). Being a function of optimal wealth (consumption) which differs across states, it is also state dependent. Figure 1 in the Appendix shows to which extent the investor demands insurance: If relative risk aversion is very high, he strives to acquire full insurance, since $1 - 1/R(\cdot)$ is close to one. If the investor has logarithmic utility (that is $R(\cdot) = 1$), he does not hedge against shocks in the investment opportunity set, since $1 - 1/R(\cdot) = 0$. This is the well known myopic result. Yet, if the investor is less risk averse than the logarithmic utility investor (that is, he has a relative risk aversion smaller than 1), he increases his exposure to the risk of changes in the investment opportunity set: The negative sign of the term $1 - 1/R(\cdot), 0 < R(\cdot) < 1$, cancels with the negative sign in the definition of $w^{(h)}(t)$ (18). We call such a demand speculative demand, but still follow the tradition to say the investor hedges against unfavorable shifts in the investment opportunity set.

The demand $w_k^{(h)}(t)$ for the k-th hedging portfolios, given by the conditional expectations in (24), is the probability weighted average of hedging demands in the different states of the world. As a conditional expectation, it is stochastic, depending on observations.

2.3.2 Constant relative risk aversion

The amounts invested into the instantaneously mean-variance efficient portfolio and in the hedging portfolios, the demands $w^{(mv)}(t)$ and $w^{(h)}(t)$, simplify in the case where the investor has constant relative risk aversion, as will be shown in this section. To carry out numerical studies later on, we need to specify a specific utility function to evaluate the expressions for the optimal wealth and consumption. We therefore specialize the results for an investor with power utility.

We assume that relative risk aversion is constant for all levels of wealth (consumption), $R_c(c) = R_B(X) = R, c, X \geq 0$. In this case, the relative risk aversion is independent of the wealth (consumption) in a given state of the world and can be factored out of the conditional expectations defining $w^{(mv)}(t)$ and $w^{(h)}(t)$. The general expression (16) for the stock portfolio for both the consumption and investment problem simplifies to

$$\pi(t) = \pi^{(mv)}(t)\frac{1}{R}X_j(t) + \pi^{(h)}(t)w_j^{(h)}(t),\ j = i, c, \tag{25}$$

where

$$X_i(t) \equiv \xi(t)^{-1}\,\mathbb{E}_t\left[\xi(T)X_i(T)\right] = \xi(t)^{-1}\,\mathbb{E}_t\left[\xi(T)I_B(y\xi(T))\right], \tag{26}$$

$$w_i^{(h)}(t) = -(1-1/R)\xi(t)^{-1}\,\mathbb{E}\left[\xi(T)X_i(T)h_t(T)|\mathcal{F}(t)\right] \tag{27}$$

for the investment problem, and

$$X_c(t) \equiv \xi(t)^{-1}\,\mathbb{E}_t\left[\int_t^T \xi(v)\hat{c}(v)dv\right] = \xi(t)^{-1}\,\mathbb{E}_t\left[\int_t^T \xi(v)I(y\xi(v))dv\right], \tag{28}$$

$$w_c^{(h)}(t) = -(1-1/R)\xi(t)^{-1}\,\mathbb{E}\left[\int_t^T \xi(v)\hat{c}(v)h_t(v)dv|\mathcal{F}(t)\right] \tag{29}$$

for the consumption problem. Thus, the demand for the instantaneously mean-variance efficient portfolio is proportional to the present wealth and the investor invests a constant fraction $\frac{1}{R}$ of his wealth into $\pi^{(mv)}(t)$.

As discussed in the previous section, depending on the relative risk aversion, the hedging demand insures against shocks ($R(\cdot) > 1$) or increases the exposure with respect to shocks ($R(\cdot) < 1$). Whereas in the general case the investor's relative risk aversion is state dependent and the total hedging demand is an average of the demands in the different states of the world, relative risk aversion is now constant and, therefore, a clear statement can be made on the investor's attitude towards risk: If his relative risk aversion is very high, he strives for acquires full insurance. If the investor has the relative risk aversion of logarithmic utility (that is $R = 1$), he does not hedge against shocks in the investment opportunity set. Finally, if the investor is less risk averse than the logarithmic utility investor, he increases his exposure to the risk of changes in the investment opportunity set.

In order to carry out numerical studies later on, we further specialize the expressions derived for the optimal consumption and wealth for the case where investors have power utility,

$$u(c,t) = \frac{\beta(t)}{1-R}c^{1-R},\ R > 0, R \neq 1, \tag{30}$$

where $\beta(t)$ is the investor's discount function, investors have constant relative risk aversion of R. The optimal terminal wealth (5) in the investment problem writes as

$$x_i(T) \equiv (y\xi(T))^{-1/R} \tag{31}$$

and equation (7), defining the Lagrange multiplier, can be solved for y,

$$y_i^{-1/R} = \frac{x}{\mathbb{E}\left[\xi(T)^\rho\right]}. \tag{32}$$

Analogously, for the consumption problem, the optimal consumption (6) becomes

$$\hat{c}(t) = \left(\frac{y\xi(t)}{\beta(t)}\right)^{-1/R}, \tag{33}$$

and (7) can be solved for y,

$$y_c^{-1/R} = \frac{x}{\mathbb{E}\left[\int_0^T \xi(t)^\rho \beta(t)^{1/R} dt\right]}, . \tag{34}$$

where we use the notation

$$\rho \equiv 1 - \frac{1}{R}. \tag{35}$$

Using these explicit expressions for the optimal wealth and the Lagrange multiplier, we find that for an investor with power utility (30), the optimal stock portfolio (25) for the investment problem simplifies to

$$\pi(t) = \frac{1}{R}(\Sigma(t)')^{-1}\theta(t)x_i(t) - \left(1 - \frac{1}{R}\right)(\Sigma(t)')^{-1}\xi(t)^{-1}m_i(t), \tag{36}$$

where

$$x_i(t) \equiv \xi(t)^{-1}\,\mathbb{E}_t\left[\xi(T)x_i(T)\right] = x\xi(t)^{-1}\,\mathbb{E}_t\left[\frac{\xi(T)^\rho}{\mathbb{E}\left[\xi(T)^\rho\right]}\right]$$

$$w_i^{(h)}(t) \equiv -(1-1/R)\xi(t)^{-1}\,\mathbb{E}_t\left[\xi(T)x_i(T)h_t(T)\right] = -(1-1/R)\xi(t)^{-1}x\,\mathbb{E}_t\left[\frac{\xi(T)^\rho}{\mathbb{E}\left[\xi(T)^\rho\right]}h_t(T)\right], \tag{37}$$

For the consumption problem, the optimal stock portfolio (25) simplifies to

$$\pi(t) = \frac{1}{R}(\Sigma(t)')^{-1}\theta(t)x_c(t) - \left(1 - \frac{1}{R}\right)(\Sigma(t)')^{-1}m_c(t) \tag{38}$$

where

$$x_c(t) \equiv \xi(t)^{-1}\,\mathbb{E}_t\left[\int_t^T \xi(v)\hat{c}(v)dv\right] = x\xi(t)^{-1}\,\mathbb{E}_t\left[\frac{\int_t^T \xi(v)^\rho\beta(v)^{1/R}dv}{\mathbb{E}\left[\int_t^T \xi(v)^\rho\beta(v)^{1/R}dv\right]}\right],$$

$$w_c^{(h)}(t) \equiv -(1-1/R)\xi(t)^{-1}\,\mathbb{E}_t\left[\int_t^T \xi(v)\hat{c}(v)h_t(v)dv\right] = -(1-1/R)\xi(t)^{-1}x\,\mathbb{E}_t\left[\frac{\int_t^T \xi(v)^\rho\beta(v)^{1/R}h_t(v)dv}{\mathbb{E}\left[\int_t^T \xi(v)^\rho\beta(v)^{1/R}dv\right]}\right].$$

The expressions found for the case of constant relative risk aversion will be used in the next section to derive the investor's trading volume. The expressions for the particular case of an agent with power utility will then be applied in numerical studies of the properties of the trading volume.

2.4 Optimal trading volume of an investor with constant relative risk aversion

In the previous section, the structure of the optimal stock portfolio was presented. It consists of a demand for the instantaneously mean-variance efficient portfolio and a demand for portfolios hedging against shocks to the investment opportunity set. Shocks to the economy will change both the structure and the demand for the portfolios and will require that the investor adjusts the portfolio and thus create trading volume. In this section, we abstract from other sources of trading, such as liquidity shocks, differences in information or beliefs, differences in the length of the investment time horizon. As throughout the paper, trading can take place at no cost and continuously. Trading volume is fully explained by the fundamental shocks to the economy that drive asset prices. We derive in this section the trading volume of a single investor who has constant relative risk aversion and maintains at all times an optimal portfolio as described in the previous section. The result derived holds in general, for any specification of the exogeneously given processes for the interest rate, market price of risk and state variables. In the next section, we will extend the analysis to the case where agents have different beliefs.

Proposition 2. The trading volume of an investor with constant relative risk aversion, facing either the investment (Problem 1) or the consumption problem (Problem 2), in the k-th stock is given by

$$\begin{aligned} d\tilde{\pi}_k(t) = {} & \frac{1}{R}\left(X_j(t)d\pi_k^{(mv)}(t) + \pi_k^{(mv)}(t)(dX_j(t) - X_j(t)dG_k(t)) + d\pi_k^{(mv)}(t)dX_j(t)\right) \\ & + \pi_k^{(h)}(t)(dw_j^{(h)}(t) - w_j^{(h)}(t)dG_k(t)) + d\pi_k^{(h)}(t)w_j^{(h)}(t) + d\pi_k^{(h)}(t)dw_j^{(h)}(t),\ j = i, c, \end{aligned} \tag{39}$$

where subscripts k denote the k-th element of vectors, or, in the case of $\pi^{(h)}(t)$ and $d\pi^{(h)}(t)$, the k-th row of a matrix, where the optimal wealth $X_j(t)$, $j = i, c$ solves

$$\begin{aligned} dX_j(t) = {} & \left[X_j(t)\left(r(t) + \frac{1}{R}\theta(t)'\theta(t)\right) + \theta(t)'\sigma^Y(t)'w_j^{(h)}(t)\theta(t)\right]dt \\ & + \left[\frac{1}{R}X_j(t)\theta(t) + \sigma^Y(t)'w_j^{(h)}(t)\right]' dW(t), \end{aligned} \tag{40}$$

the demand for the k-th hedging portfolio, $k = 1, 2, \ldots, p$, solves

$$dw_{j,k}^{(h)}(t) = \left[w_{j,k}^{(h)}(t)\left(r(t) + \theta(t)'\theta(t)\right) - \rho\xi(t)^{-1}\theta(t)'\tilde{M}_{j,k}(t)\right] dt + \left[w_{j,k}^{(h)}(t)\theta(t)' - \rho\xi(t)^{-1}\tilde{M}_{j,k}(t)\right] dW(t), \tag{41}$$

where

$$\tilde{M}_{i,k}(t) = \mathbb{E}_t\left[\xi(T)X_i(T)\left[-(1-1/R)(\theta(t) + \sigma^Y(t)'h_t(T))h_{k,t}(T) + \mathcal{D}_t\left(h_{k,t}(T)\right)\right]\right], \tag{42}$$

$$\tilde{M}_{c,k}(t) = \mathbb{E}_t\left[\int_t^T \xi(v)\hat{c}(v)\left[-(1-1/R)(\theta(t) + \sigma^Y(t)'h_t(v))h_{k,t}(v) + \mathcal{D}_t\left(h_{k,t}(v)\right)\right] dv\right], \tag{43}$$

the mean variance portfolio solves

$$d\pi^{(mv)}(t) = \left[\bar{\Sigma}(t)\mu^\theta(t) + \mu^{\bar{\Sigma}}(t)\theta(t) + \sigma^{\bar{\Sigma}}(t)'\sigma^\theta(t)\right] dt + \left[\bar{\Sigma}(t)\sigma^\theta + \sigma^{\bar{\Sigma}}(t)\theta(t)\right] dW(t) \tag{44}$$

the (i, j) element of the hedging portfolio solves

$$d\pi_{i,j}^{(h)}(t) = \sum_{k=1}^{d}\left\{\left[\bar{\Sigma}_{ik}(t)\mu_{kj}^{\sigma^Y}(t) + \sigma_{kj}^Y(t)\mu_{ik}^{\bar{\Sigma}} + \frac{1}{2}\sigma_{ik}^{\bar{\Sigma}}(t)\sigma_{kj}^{\sigma^Y}(t)\right] dt + \left[\bar{\Sigma}_{ik}\sigma_{kj}^{\sigma^Y} + \sigma_{kj}^Y(t)\sigma_{ik}^{\bar{\Sigma}}(t)\right] dW_k(t)\right\}. \tag{45}$$

Proof. See the Appendix. □

$d\tilde{\pi}_k(t)$ is the investor's instantaneous trading volume in the k-th stock expressed 'in dollars', that is, the amount of money traded during an infinitesimal amount of time in this stock. This trading volume is optimal in the sense that it results from pursuing the optimal trading strategy. It is the investor's reaction to changes in the investment opportunity set due to the shocks $dW(t)$ to the economy.

Proposition 2 shows that the volume is driven by the following quantities. The change of the composition of the instantaneously mean-variance efficient portfolio, $d\pi^{(mv)}(t)$; the change in wealth $dX_j(t)$; the change in the optimal amounts to be invested into the different hedging portfolios, $dw^{(h)}(t)$; the change of the composition of the hedging portfolios, $d\pi^{(h)}(t)$; and the gains on the position in the k-th stock, $dG_k(t)$ (see (1)). Due to the Markovian structure of the economy on one hand and the investor's time-additive utility function on the other hand, these quantities are again Ito processes. Consequently, $d\tilde{\pi}(t)$ being a linear combination of Ito processes, the trading volume belongs as well to this class of processes. Its drift and diffusion coefficients will be studied in a numerical analysis in the next section.

The trading volume $d\tilde{\pi}_k(t)$ (39) in the k-th stock, $k = 1, 2, \ldots, d$, can be attributed to six different motives. The following three are related to the mean-variance efficient portfolio:

- Due to shocks $W(t)$ to the economy, the investor needs to adapt the amount of money that he has invested into the mean-variance efficient portfolio. Therefore, he trades in the k-th component stock an amount equal to

$$\frac{1}{R}\pi_k^{(mv)}(t)(dX_j(t) - X_j(t)dG_k(t)),\ j = i, c.$$

  This is due, on one hand, to the change of the investor's wealth. To maintain a fraction of $\frac{1}{R}$ of his wealth to be invested in the mean-variance efficient portfolio, he needs to trade $\frac{1}{R}\pi^{(mv)}(t)dX_j(t)$. On the other hand, the investor earns a return of $\frac{1}{R}\pi_k^{(mv)}(t)X_j(t)dG_k(t)$ on the position in the k-th stock in the mean-variance portfolio $\pi^{(mv)}(t)$. The investor needs to trade the difference of the required rebalancing and the earned return. Risk aversion affects the trading volume in the instantaneously mean-variance efficient portfolio $\pi^{(mv)}(t)$ in multiple ways, therefore having an ambiguous effect: The dynamics of the optimal wealth $dX_j(t)$ (40) depend on the hedging demand $w^{(h)}(t)$, and this depends on the optimal wealth (see (27)) for the investment problem and on the optimal consumption (see (29)) for the consumption problem, respectively. These optimal quantities will in general be influenced by the relative risk aversion, as it is the case for the power utility in the expressions for optimal wealth (31) and consumption (33).

- The shocks $W(t)$ to the economy also have an impact on the expected rate of return $\mu(t)$ and the diffusion matrix $\Sigma(t)$ of the risky assets as well as the interest rate $r(t)$. Because the structure of the instantaneously mean-variance efficient portfolio $\pi^{(mv)}(t)$ (15) depends on these quantities, the investor needs to adjust its holdings to the new target structure and thus trades an amount

$$\frac{1}{R}X_j(t)d\pi_k^{(mv)}(t),\ j = i, c.$$

  in the k-th stock.

- A further component of the trading volume stems from the covariation of the instantaneously mean-variance efficient portfolio and the optimal wealth,

$$d\pi^{(mv)}(t)dX_j(t)).$$

Three other sources of trading volume are related to the hedging component of the investor's stock demand:

- Due to the shocks $W(t)$ to the economy, the optimal amount that the investor wishes to have invested into the hedging portfolios changes. Adapting the weights of each of these hedging portfolios in his total stock portfolio lets the investor trade an amount of

$$\pi_k^{(h)}(t)(dw_j^{(h)}(t) - w_j^{(h)}(t)dG_k(t)),\ j = i, c$$

  in the k-th stock. On one hand, his demand for the p hedging portfolios changes, due to the change of his information set upon observing the current shocks to the economy, $W(t)$. Based on this observation, the level of his expected future optimal wealth (consumption) as well as the sensitivity $h_t(v)$ of the future investment opportunity set changes. Thus, the optimal demand for the hedging portfolios, $w_j^{(h)}(t)$, changes. Adapting the weights of the hedging portfolios in his total stock portfolio lets the investor trade $\pi_k^{(h)}(t)dw_j^{(h)}(t)$ in the k-th stock.[4]At the same time, he earns on the total holdings of the k-th stock in his hedging portfolio a return of $\pi_k^{(h)}(t)w_j^{(h)}(t)dG_k(t)$. Thus, his trade in the k-th component $\pi_k^{(h)}(t)$ of the hedging portfolio equals the difference of the required rebalancing and the return earned on this portfolio position.

- The structure of the hedging portfolios $\pi^{(h)}(t)$ (17) is also changed by the shocks $W(t)$ and this requires from the investor that he rebalances the hedging portfolio. He adapts the composition of each of the p hedging portfolios such that the portfolios have again the maximal correlation with the state variables in the changed market, thus trading an amount

$$d\pi_k^{(h)}(t)w_j^{(h)}(t),\ j = i, c$$

  in the k-th stock.

- And, lastly, there is a component of the trading volume due to the covariation of the hedging portfolios and the optimal amounts invested into them,

$$d\pi^{(h)}(t)dw_j^{(h)}(t).$$

Several points regarding the trading volume derived are worth to be noted. In the theoretical literature on trading volume in discrete time as well as in

[4]Observe that $\pi_k^{(h)}(t)$ as the k-th row of the matrix $\pi^{(h)}(t)$ is a row vector, where $dw_j^{(h)}(t)$ is a column vector.

empirical work, the cumulated trading volume plays an important role. The instantaneous trading volume $d\tilde{\pi}(t)$ can be positive or negative. It would be interesting to have a notion of absolute trading volume. This could be cumulated across investors. Further, it could be cumulated over time. The trading volume, cumulated over time, plays an important role in models using a time-change to model return distributions (see e.g. Ané and Geman (2000)).

Yet, in any setting where shocks to the economy are modelled as Brownian motions and investors can trade continuously, trading volume will be of unbounded variation over any time interval. This reflects the investors' reactions to the continuously changing information set, due to the movement in the Brownian motion. Only the quadratic variation will be finite. This is definitely a shortcoming of this approach of modelling trading volume starting from a diffusion process.

2.5 A numerical procedure to approximate trading volume

Expression (39) found for the trading volume can not easily be analyzed analytically. We therefore develop in this section a numerical procedure to approximate the drift and volatility of trading volume. This numerical procedure facilitates any analysis of the properties of the trading volume. Its use will be illustrated by applying it to the economic setting of Detemple et al. (2003) who have explored the properties of investors' optimal portfolio. Thereafter, it will be used to analyze in the next section the effect of heterogeneity of beliefs and informational incompleteness on trading volume.

The main difficulty in approximating the expressions for the trading volume is the presence of the sensitivity $h_t(v)$ of the investment opportunity set with respect to the state variables and its Malliavin derivative $\mathcal{D}_t\left(h_t(v)\right)$. Both the demand for the hedging portfolios $w^{(h)}(t)$ in (18) as well as the terms $\tilde{M}_{jk}(t)$, $j = i, c$ in (42) and (43) depend on these quantities. The numerical method used in this paper to find an estimator of the terms mentioned builds on Detemple et al. (2003) who have analyzed the optimal portfolio of an investor as described by equation (8). Therefore, they had to find an approximation of the Malliavin derivatives of the interest rate and the market price of risk in $H_t(v)$ in (14). They reduced the problem to the approximation of the Malliavin derivatives of the state variable, due to the equalities[5]

$$\mathcal{D}_t\left(r(u)\right) = \partial_2 r(u)\mathcal{D}_t\left(Y(u)\right), \tag{46}$$

$$\mathcal{D}_t\left(\theta(u)\right) = \partial_2 \theta(u)\mathcal{D}_t\left(Y(u)\right). \tag{47}$$

[5]This follows from the fact that $r(t,Y)$ and $\theta(t,Y)$ are smooth functions.

In the current paper, the Malliavin derivatives have been substituted by the matrix process $\Phi_t(v)$,

$$\mathcal{D}_t\left(Y(v)\right) = \Phi_t(v)\sigma^Y(t, Y(t)) \tag{48}$$

(this is (80) in the proof of Proposition 1) where the $(d \times d)$ matrix $\Phi_t(v)$ is the solution of the system (22) of vector SDE's

$$\begin{aligned} d\Phi_t(u) &= \partial_2\mu^Y(u)\Phi_t(u)du + \sum_{j=1}^{d} \partial_2\sigma^Y_{\cdot j}(u)\Phi_t(u)dW_j(u), \\ \Phi_t(t) &= I. \end{aligned} \tag{22}$$

This system of SDE's lends itself for approximation by the Euler scheme. To approximate $h_t(v)$, as given by equations (19) to (21), the approximation of $\Phi_t(u)$ is used together with analogous approximations of $\partial_2 r(u), \theta(u)$ and $\partial_2\theta(u)$ which are all solutions to SDE's. The approximation of the trading volume then further requires to find an approximation of the Malliavin derivative $\mathcal{D}_t\left(h_t(v)\right)$ of $h_t(v)$. Applying the chain rule of Malliavin calculus to $h_t(v)$ as defined in (19) yields that also $\mathcal{D}_t\left(h_t(v)\right)$ is the solution to a system of SDE's,

$$\begin{aligned} \mathcal{D}_t\left(h_t(v)'\right) = &\int_t^v \mathcal{D}_t\left(\partial_2 r(u)\Phi_t(u)\right)du + \int_t^v dW(u)'\mathcal{D}_t\left(\partial_2\theta(u)\Phi_t(u)\right) \\ &+ \int_t^v \mathcal{D}_t\left(\theta(u)\partial_2\theta(u)\Phi_t(u)\right)du)' + \partial_2\theta(t),\ v \geq t. \end{aligned}$$

The coefficients of this system of SDE's, the product of $\Phi_t(u)$ with derivatives of $r(u)$ and $\theta(u)$ respectively, involve further Malliavin derivatives. Application of the chain rule of Malliavin calculus to these coefficients yields expressions involving the Malliavin derivative of $\Phi_t(u)$. $\Phi_t(u)$ is defined as the solution of the system (22) of linear stochastic differential equations and the coefficients in these SDE's are assumed to be continuous and to satisfy growth conditions. Therefore, Theorem 2.2.1 in Nualart (1995) implies that $\mathcal{D}_t\left(\Phi_t(u)\right)$ belongs to $\mathbb{D}^{1,\infty}$ for all $u \in [0, T]$; $\mathcal{D}_t\left(\Phi_t(u)\right)$ is thus well defined.

Hence, we find that the Malliavin derivative $\mathcal{D}_t\left(h_t(v)\right)$ of $h_t(v)$ is also the solution of a system of SDE's and can therefore be approximated by using the Euler scheme. In addition, one has also to simulate the Malliavin derivative $\mathcal{D}_t\left(\Phi_t(v)\right)$ of $\Phi_t(v)$. The SDE's that these two quantities solve are given in Lemma 5 in the Appendix. For the investment problem (to save space, we don't print the analogous quantities for the consumption problem; these will be used in the second part of the paper in an economy with heterogeneous beliefs), this leaves us with the following system of SDE's to be simulated:

$$d\xi(u) = -\xi(u)(r(u)du + \theta(u)'dW(u)) \tag{49}$$

$$d\Phi_0(u) = \partial_2\mu^Y(u)\Phi_0(u)du + \sum_{j=1}^{d}\partial_2\sigma^Y_{\cdot j}(u)\Phi_0(u)dW_j(u), \tag{50}$$

$$dh_0(u)' = \partial_2 r(u)\Phi_0(u)du + (dW(u) + \theta(u)du)'\partial_2\theta(u)\Phi_0(u) \tag{51}$$

$$d\mathcal{D}_{j,0}\left(h_{i,0}(u)\right) = \ldots \text{given in Lemma 5 in the Appendix,} \tag{52}$$

$$d\mathcal{D}_{n,0}\left(\Phi_{i,j,0}(u)\right) = \ldots \text{given in Lemma 5 in the Appendix.} \tag{53}$$

M paths of the solutions of these equations are simulated simultaneously, using the Euler scheme with a time discretization into N points. This yields M estimates $\xi^m(T)$, $\Phi^m_t(T)$, $h^m_t(T)$, $\mathcal{D}^m_{j,t}\left(h_{i,t}(T)\right)$, $\mathcal{D}^m_{n,t}\left(\Phi_{i,j,t}(T)\right)$. Averaging over the different sample paths yields the approximation of the demand (37) for the hedging portfolios

$$\hat{w}^{(h)}(0) = -(1-1/R)x\sum_{m=1}^{M}\frac{\xi^m(T)^\rho h^m_t(T)}{\sum_{m=1}^{M}\xi^m(T)^\rho}. \tag{54}$$

Further, an estimator of $\tilde{M}_{i,k}(0)$, defined in (42) is provided by

$$\hat{M}_{i,k}(0) = \sum_{m=1}^{M}\frac{(\xi^m(T))^\rho\left[-(1-1/R)(\theta(0)+\sigma^Y(0)'h^m_0(T))h_{k,0}(T) + \mathcal{D}^m_0\left(h_{k,0}(T)\right)\right]}{\sum_{m=1}^{M}\xi^m(T)^\rho}. \tag{55}$$

Together with the exogenous parameters this suffices to calculate the dynamics of the trading volume (39) as stated in Proposition 2.

2.6 Trading volume of an investor with complete information

The numerical method presented in the previous section allows us to calculate an approximation of the trading volume. In order to make quantitative statements on the different components of the trading volume, we need to choose a particular economic setting for the interest rate and market price of risk. We choose the setting proposed by Detemple et al. (2003) who have studied the properties of the optimal portfolio. We will thereby add to their analysis the aspect of the dynamics of the portfolio, how the the investor adapts his positions by trading.

The approximation presented depends on a specification of the following exogenous quantities: The investor's preferences, as captured by this relative risk aversion parameter, the dynamics (2) of the state variables $Y(t)$ and the function $r(t, Y)$ and $\theta(t, Y)$. Detemple et al. (2003) have chosen as state variables the interest rate $r(t)$ and the market price of risk $\theta(t)$. As for the dynamics of these quantities, they propose the following processes: The interest rate solves

$$dr(t) = \kappa_r(\bar{r} - r(t))\left(1 + \phi_r(\bar{r} - r(t))^{2\eta_r}\right)dt - \sigma_r r(t)^{\gamma_r} dW(t), \quad r(0)\,\text{given},$$

and the market price of risk solves

$$d\theta(t) = \left(\kappa_\theta(\bar{\theta} - \theta(t)) + \mu_\theta^r(r(t), \theta(t))\right)dt + \sigma_\theta(\theta(t))dW(t), \quad \theta(0)\,\text{given},$$

where

$$\mu_\theta^r(r(t), \theta(t)) = \delta_r(\bar{r} - r(t))(\theta_l + \theta(t))\left(1 - \frac{\theta_l + \theta(t)}{\theta_l + \theta_u}\right),$$

$$\sigma_\theta(r(t), \theta(t)) = \sigma_\theta(\theta_l + \theta(t))^{\gamma_{1\theta}}\left(1 - \left(\frac{\theta_l + \theta(t)}{\theta_l + \theta_u}\right)^{1-\gamma_{1\theta}}\right)^{\gamma_{2\theta}}.$$

The term $\mu_\theta^r(r(t), \theta(t))$ captures the dependence on the interest rate of the market price of risk. The coefficients $(\kappa_r, \bar{r}, \phi_r, \eta_r, \sigma_r, \gamma_r, \kappa_\theta, \bar{\theta}, \eta_\theta, \sigma_\theta, \theta_l, \theta_u, \gamma_{1\theta}, \gamma_{2\theta})$ are constants, $(\kappa_r, \bar{r}, \kappa_\theta, \theta_l, \theta_u,)$ are positive and $\bar{\theta} \in (-\theta_l, \theta_u)$. The Brownian motion W is unidimensional. Thus, this is a model where the number of sources of uncertainty equals 1, $d = 1$, and there are two state variables, $p = 2$. The process for the interest rate incorporates mean reversion with constant elasticity of variance and exhibits nonlinearities in the speed of mean reversion, captured by $\phi_r(\bar{r} - r(t))^{2\eta_r}$, thus taking important empirical findings into account (Detemple et al., 2003). The market price of risk also exhibits mean reversion; the term μ_θ^r captures the interest rate dependence of the market price of risk and is such that the market price of risk stays between the two bounds $-\theta_l$ and θ^u. The elasticity of variance of this process is hyperbolic. These two specifications therefore take nonlinearities in the mean and variance into account which have been found in the data (Detemple et al., 2003).

Detemple et al. (2003) have estimated σ_r and found a positive value. This implies together with positive stock volatility and the fact that there is only one Brownian motion that the stock and interest rate are negatively correlated. On the other hand, $\sigma_\theta(r, \theta) > 0$ for all $r \geq 0$ and $\theta \in (-\theta_l, \theta_u)$; therefore, the market price of risk and the stock are positively correlated.

The values of the parameters used for the simulation study are given in Appendix B. In the current context, the investor is fully described by his relative risk aversion parameter and the length of his investment time horizon. In all graphs, the relative risk aversion R varies between 0.75 and 6 and the investment time horizon T varies between 1 and 10 years, that is, N varies between 52 and 520. A total of $M = 10'000$ paths has been simulated for weekly time increments. Unfortunately, we don't have error bounds for the statistical and numerical errors of the analysis. This is definitely a serious shortcoming. However, at the chosen values of N and M, computation already consumes a considerable amount of time; a further increase of the number of paths to $M = 25'000$ changes the results only marginally, therefore $M = 10'000$ was chosen. The estimators (54) and (55) are calculated and used for the calculation of the portfolio (25) and expected trading volume and its volatility, as stated below in Lemma 1.

Figure 2 displays the stock demands as a function of the relative risk aversion and investment time horizon, as they have been discussed in Detemple et al. (2003). Figure 2(b) displays the total demand $\pi_i(0)$ for the single stock in the case of the investment problem, that is, the sum of the mean-variance portfolio and the hedging portfolio. As expected, the more risk averse an investor is, the smaller is his stock position. Also, a longer investment time horizon corresponds to a larger investment into the risky asset. The total hedging stock demand $\pi^{(h)}(0)w_i^{(h)}(0)$ is displayed in Figure 2(a). The relative importance of the hedging stock demand as a fraction of the total stock demand varies with risk aversion and time horizon. For example, for a relative risk aversion close to 1, the investor does not hedge and all the stock demand comes from the demand for the mean-variance efficient portfolio only. On the other hand, when the investment time horizon is long and the investor has a high degree of relative risk aversion, most of his demand for the stock stems from the hedging stock demand (see also Brennan et al. (1997), Detemple et al. (2003)).

The diffusion process governing the trading volume can be characterized by its drift and diffusion term, as given in the following

Lemma 1. The trading volume in the k-th stock has a drift of

$$\begin{aligned}\mathbb{E}\left[d\tilde{\pi}_k(t)\right] =& \frac{1}{R}X_j(t)\,\mathbb{E}\left[d\pi_k^{(mv)}(t)\right] + \frac{1}{R}\pi_k^{(mv)}(t)(\mathbb{E}\left[dX_j(t)\right] - X_j(t)\mu_k(t)dt) + \frac{1}{R}V(d\pi_k^{(mv)}(t))V(dX_j(t)))\\ &+\mathbb{E}\left[d\pi_k^{(h)}(t)\right]w_j^{(h)}(t) + \pi_k^{(h)}(t)\left(\mathbb{E}\left[dw_j^{(h)}(t)\right] - w_j^{(h)}(t)\mu_k dt\right) + V(d\pi_k^{(h)}(t))V(dw_j^{(h)}(t)),\; j=i,c,\\ &=: EMVREBAL_{k,j}(t) + EMVTRAD_{k,j}(t) + COVMV_{k,j}(t)\\ &\quad + EHREBAL_{k,j}(t) + EHTRAD_{k,j}(t) + COVH_{k,j}(t)\end{aligned}$$

and a diffusion term of

$$
\begin{aligned}
V(d\tilde{\pi}_{k,j}(t)) = & \frac{1}{R}X_j(t)V(d\pi_k^{(mv)}(t)) + \frac{1}{R}\pi_k^{(mv)}(t)(V(dX_j(t)) - X_j tV(dG_{k,j}(t))) \\
& + \pi_k^{(h)}(t)(V(dw_j^{(h)}(t)) - w_j^{(h)}(t)'V(dG_{k,j}(t))) + w_j^{(h)}(t)V(d\pi^{(h)}(t)) \\
=: & VMVREBAL_{k,j}(t) + VMVTRAD_{k,j}(t) \\
& + VHREBAL_{k,j}(t) + VHTRAD_{k,j}(t)
\end{aligned}
$$

where the notation $V(dZ(t)) = B(t)$ for a diffusion of the general type $dZ(t) = A(t)dt + B(t)dW(t)$ with drift $A(t)$ and diffusion term $B(t)$ is used.

Proof. See the Appendix. □

The three terms in the first line of (56) are the expected volume that comes from the mean-variance component of the investor's stock demand, the three terms in the first line of (56) are the corresponding volatility of volume, stemming from the mean-variance component. The three terms in the second line of (56) are the expected volume that comes from the hedging component of the investor's stock demand, the three terms in the second line of (56) are the corresponding volatility of volume.

Figure 3 displays approximations of both moments, $\mathbb{E}\left[d\tilde{\pi}_k(t)\right]$ and $V(d\tilde{\pi}_k(t))$, that result for different values of the relative risk aversion R and time horizon T. Both show little effect of the length of the investment horizon, the effect of the risk aversion dominates. For increasing risk aversion, the expected volume approaches zero; when risk aversion approaches zero, expected trading volume increases in absolute size.

The reason for this behavior becomes clear from the components of the expected volume, shown in Table 2 for a risk aversion of 0.75, 1 and 3 and investment time horizon ranging from 1 to 10 years. The expected trading volume coming from the mean-variance component dominates, the largest part stemming from rebalancing the mean-variance portfolio, EMVREBAL. Since a more risk averse investor invests less money into the mean-variance efficient portfolio (see Figure 2(b)), the absolute size of his trading volume from the the mean-variance component is also smaller. Since the other components of the expected trading volume are of smaller magnitude and tend to cancel out due to different signs, the expected volume EMVREBAL from rebalancing the mean-variance portfolio dominates the total expected volume. Therefore, the time horizon has little effect on expected volume EVOLUME because the structure of the mean-variance portfolio is independent of the investor's preferences or time horizon. Both the expected volume stemming from adjusting the weight of the mean-variance efficient portfolio in the investor's total portfolio, EMVTRAD, as well as the covariation COVMV decrease in absolute size with increasing risk aversion. This property is common to all mean-variance related components of the expected trading volume and reflect the smaller investment in the mean-variance portfolio of more risk-adverse investors.

Expected trading volume due to hedging corresponds to the three terms in the second line of (56). The expected volume EHREBAL from rebalancing the hedging portfolio, shown in Table 2, is small and increases in absolute size with the length of the time horizon. This is due to the fact that investors with a longer time horizon demand a larger amount of the hedging portfolios, see the total hedging demand in Figure 2(a). Therefore, they need to trade a larger amount to adapt the structure of the hedging portfolio to the current value.

A further result from Figure 3 is that the hedging demand changes sign at a relative risk aversion of 1; this is formally shown in

Lemma 2. Expected trading volume and volatility of trading volume stemming from the hedging portfolio equals zero for a risk aversion of 1, and changes sign at a risk aversion of 1.

Proof. See the Appendix. □

While it is a trivial consequence of a logarithmic utility investor's myopia that his expected trading volume and volatility equal zero, the change in sign is an interesting new result. The 'knife-edge behavior of logarithmic utility' (Detemple et al., 2003, p. 415) is also observed for the expected trading volume and its volatility. Column EHVOLUME of Table 2 illustrates this theoretical result for the case of the Detemple et al. (2003) benchmark economy.

Table 3 shows the volatility of volume, both for a risk aversion of 0.75, 1 and 3 and investment time horizon ranging from 1 to 10 years. Volatility VMVVOLUME of trading due to the mean-variance component decreases when risk aversion increases. Thus, risk aversion reduces both the expected size of trading volume related to the mean-variance portfolio as well as its volatility. As stated in Lemma 2, the volatility due to hedging also changes sign at a risk aversion of 1. Whenever the relative risk aversion is different from 1, the hedging motivation contributes an important part to the volatility of trading volume, opposed to its expected value where the mean-variance component dominated for most values of the relative risk aversion and investment time horizon. It becomes quite large in absolute size when risk aversion increases. This is mainly due to the volatility from adjusting the amounts invested into the hedging portfolios, VHTRAD. Thus, trading due to hedging of more risk averse investors is more volatile because they adjust their holdings of the hedging portfolios more often. Again, the time horizon has only a small effect on the volatility from the mean-variance component (since the structure of the mean-variance portfolio is independent of the time horizon, it has no effect on the volume VMVREBAL stemming from rebalancing the structure of this portfolio, yet it has a small effect on VMVTRAD since this volume is caused by the change in wealth). A longer time horizon increases the volatility VHVOLUME related to the hedging portfolio in absolute size.

For the consumption problem, the analogous quantities on the portfolio structure and the trading volume look very similar and are therefore not shown here. Yet, the demand for the interest rate hedging portfolio $\pi_1^{(h)}(0)$ is much higher than before.

Summarizing, the largest part of the expected volume stems from rebalancing the mean-variance portfolio. Therefore, time horizon has little effect on expected volume. Since a more risk averse investor invests less money into the mean-variance efficient portfolio, both his expected trading volume and its volatility are lower than for a less risk averse investor. Yet, trading volume related to the hedging portfolio is more volatile for investors with a higher risk aversion. Also a longer investment horizon increases the volatility of trading volume. Both the expected volume from the hedging portfolio as well as its volatility change sign at a risk aversion of 1.

A clear limitation of the above results is that frictionless trading opportunities are assumed. In the presence of trading costs, be it fixed or proportional, the result would clearly be quite different. Further, it would be interesting to take the above conclusions to data on the behavior of traders. Testable hypotheses could be formulated in terms of risk aversion and time horizon. This remains for further research.

3 The impact of heterogeneous beliefs on trading volume

We now turn to the analysis of the effect that heterogeneity of beliefs and incompleteness of information have on trading volume in equilibrium. The expressions found in the previous sections for the trading volume as well as the numerical procedure proposed are general enough to be applied to the current economic setting as well.

The model presented in this section is a pure exchange economy where agents have incomplete information on the expected growth rate of aggregate consumption. Investors all observe realized growth rates, thus having symmetric, but incomplete information. Because they do not know the true initial value of the expected growth rate but perceive it to be random, they cannot disentangle the expected value and the stochastic component of growth rates. The agents have heterogeneous beliefs about the unobservable growth rate. The model follows Detemple and Murthy (1994) who looked at a production economy and the related papers by Zapatero (1998), Basak (2000) and Berrada (2002) who looked at pure exchange economies with incomplete information.

3.1 Structure of the economy, information and beliefs

Aggregate consumption is given exogenously by

$$dD(t) = (A_0(t, D(t)) + A_1(t, D(t))\gamma(t))dt + B(t, D(t))dW_1(t), \quad D(0) = D_0, \tag{56}$$

where the component $\gamma(t)$ of the growth rate is the solution of

$$d\gamma(t) = (a_0(t) + a_1(t)\gamma(t))dt + b(t)dW_0(t), \quad \gamma(0) = \gamma_0. \tag{57}$$

The specifications (56) and (57) admit a large class of processes, allowing nonlinearity in the drift and diffusion term of the aggregate consumption. The functionals $A_0(t, x)$, $A_1(t, x)$, $B(t, x)$ as well as $a_0(t, x)$, $a_1(t, x)$ and $b(t, x)$ satisfy Lipschitz and growth conditions and $B(t, x)$ is in its second argument a function bounded away from 0. $\mathbb{W} = (\mathbb{W}_0, \mathbb{W}_1)' = \{W(t)\}_{0\le t\le T}$ is a two dimensional standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The filtration $\{\mathcal{F}(t)\}_{0\le t\le T}$ is defined as the one generated by the Brownian motion $\mathbb{W}$, $\mathcal{F}(t) = \mathcal{F}^{\mathbb{W}}(t)$, $0 \le t \le t$, and is assumed to satisfy the usual assumptions[6]. The initial values γ_0 and D_0 are random variables independent of the Brownian motion $\mathbb{W}$. γ_0 is assumed to be a $\mathcal{A}^\gamma - \mathcal{B}$ measurable map $\gamma_0 : \Omega \to \mathbb{R}$ where

[6]Throughout the paper, the notation $\mathcal{F}^{\mathbb{A},\mathbb{B}}(t)$ denotes the sigma algebra generated by the stochastic processes $\mathbb{A} = \{A(t)\}_{0\le t\le T}$ and $\mathbb{B} = \{B(t)\}_{0\le t\le T}$, $\mathcal{F}^{\mathbb{A},\mathbb{B}}(t) = \sigma(A(s), B(s) : s \le t)$.

the sigma algebra $\mathcal{A}^\gamma$ is independent of $\mathcal{F}(T)$ and $\mathcal{B}$ is the Borel sigma algebra. D_0 is assumed to be a $\mathcal{A}^D - \mathcal{B}$ measurable map $D_0 : \Omega \rightarrow \mathbb{R}$ where the sigma algebra $\mathcal{A}^D$ is independent of $\mathcal{F}(T)$. Further, we assume that the initial value γ_0, conditional on the initial dividend D_0 is normally distributed with mean $\hat{\gamma}_0$ and variance $\hat{V}_0$, that is, $F_{\gamma_0}(a) \equiv \mathbb{P}(\gamma_0 \leq a|D_0)$ is almost surely $\mathcal{N}(\hat{\gamma}_0, \hat{V}_0)$.

This economy is populated by two investors, each with a utility function with constant relative risk aversion,

$$u_i(c,t) = \frac{\beta_i(t)}{1-R} c^{1-R},\ R > 0, R \neq 1, \quad i = 1, 2.$$

The utility function includes a time varying discount factor $\beta_i(t) = \exp\left(-\int_0^t \zeta_i(s)ds\right)$ where $\zeta_i(s) \geq 0, 0 \leq s \leq T$. Notice that both investors are assumed to have the same Arrow-Pratt measure of relative risk aversion, equalling R. This assumption restricts heterogeneity among agents to stem from information and initial endowment only (the agents initial wealths will be introduced below), not from preferences. It will allow us to solve for the equilibrium processes later on.

Both investors know the functional form (56) of the dynamics of $D(t)$ and of the growth rate $\gamma(t)$ in (57) as well as all the coefficients in the two SDE's. They can only observe the path of aggregate consumption $D(t)$, but not the growth rate $\gamma(t)$. Their information set, $\mathcal{F}^D(t)$ is strictly smaller than the full information set $\mathcal{F}(t)$. Thus, agents have homogeneous incomplete information, yet they have heterogeneous beliefs represented by probability measures $\mathbb{P}^i$, $i = 1, 2$ which are both equivalent to the true probability measure $\mathbb{P}$. The measures $\mathbb{P}^1, \mathbb{P}^2$ and $\mathbb{P}$ coincide everywhere except on $\mathcal{A}^\gamma \otimes \mathcal{A}^D$. Investor $i, i = 1, 2$, perceives the initial value γ_0, conditional on the initial aggregate endowment D_0, to be normally distributed with mean $\gamma_0^{(i)}$ and variance $V_0^{(i)}$, that is, $F^{(i)}_{\gamma_0}(a) \equiv \mathbb{P}^i(\gamma_0 \leq a|D_0)$ is almost surely $\mathcal{N}(\gamma_0^{(i)}, V_0^{(i)})$. The means $\gamma_0^{(i)}$ and variances $V_0^{(i)}$ of the conditional distributions are exogenously given. The two investors can be thought to represent two groups in the economy that have different opinions on the economic perspectives. This specification of heterogeneity comprises several cases:

- When both agents agree on the conditional distribution of γ_0, that is when $\gamma_0^{(1)} = \gamma_0^{(2)}$ and $V_0^{(1)} = V_0^{(2)}$, then beliefs are not heterogeneous any more. These homogeneous beliefs might still be wrong, that is, either the perceived mean or variance deviates from the true parameters. Then, both investors agree erroneously.
- When both investors perceive the conditional distribution of γ_0 to have the same variance, $V_0^{(1)} = V_0^{(2)}$, then both have the same amount of uncertainty about the true parameter. They disagree only about the level

of γ_0. If one investor perceives the initial growth rate to be higher than the other investor, we call the former optimist and the latter pessimist.

- When both investors agree on the mean of the conditional distribution of γ_0, $\gamma_0^{(1)} = \gamma_0^{(2)}$, but have different levels of uncertainty about this mean, that is, they disagree about the variance $\hat{V}_0$, then the agent with the lower variance has more precise information about the unobservable parameter. This is then a case of incomplete and asymmetric information.

For investor $i, i = 1, 2$, the optimal estimate of the unobservable parameter $\gamma(t)$ is $\mathbb{E}_{\mathbb{P}^i}\left[\gamma(t)|\mathcal{F}^D(t)\right] =: \gamma_i(t)$. Based on this estimate, investor i makes a normalized estimation error of

$$d\nu_i(t) \equiv \frac{dD(t) - (A_0(t) + A_1(t)\gamma_i(t))dt}{B(t)} = \frac{A_1(t)(\gamma(t) - \gamma_i(t))dt}{B(t)} + dW_1(t),\ 0 \leq t \leq T. \tag{58}$$

$\{\nu_i(t)\}_{0\leq t\leq T}$ is called the innovation process. Detemple and Murthy (1994) show that $\{\nu_i(t), \mathcal{F}^D(t)\}_{0\leq t\leq T}$ is a $\mathbb{P}^i -$ Brownian motion and that the filtration, generated by $\{\nu_i(t)\}_{0\leq t\leq T}$ equals the filtration generated by the observed consumption $\{D(t)\}_{0\leq t\leq T}$. According to Theorem 12.1 in Liptser and Shiryaev (2001), $\gamma_i(t)$ solves the SDE

$$\begin{aligned} d\gamma_i(t) &= (a_0(t) + a_1(t)\gamma_i(t))dt + \frac{V_i(t)A_1(t)}{B(t)}d\nu_i(t), \\ \gamma_i(0) &= \mathbb{E}_{\mathbb{P}^i}\left[\gamma(0)|D(0)\right] = \gamma_0^{(i)}, \end{aligned} \tag{59}$$

where $\nu_i(t)$ is the innovation process (58). The variance of the estimate $V_i(t) \equiv \mathbb{E}_{\mathbb{P}^i}\left[(\gamma(t) - \gamma_i(t))^2|\mathcal{F}^D(t)\right]$ solves the ODE

$$\begin{aligned} \dot{V}_i(t) &= 2a_1(t)V_i(t) + b(t)^2 - \left(\frac{V_i(t)A_1(t)}{B(t)}\right)^2, \\ V_i(0) &= \mathbb{E}_{\mathbb{P}^i}\left[(\gamma(0) - \gamma_i(0))^2|D(0)\right] = V_0^{(i)}. \end{aligned} \tag{60}$$

Using the innovation process $\nu_i(t)$ and the estimate $\gamma_i(t)$, investor i perceives the dynamics of the aggregate consumption as

$$dD(t) = (A_0(t, D(t)) + A_1(t, D(t))\gamma_i(t))dt + B(t, D(t))d\nu_i(t), \quad D(0) = D_0.$$

3.2 The financial market

There is a riskless investment opportunity, yielding a return of $r(t)$ where $\{r(t)\}_{0\leq t\leq T}$ is $\{\mathcal{F}^D(t)\}_{0\leq t\leq T}$ adapted. There is one unit of a stock in the economy. The owner of the stock receives as a dividend the aggregate consumption $D(t)$. The dynamics of the gains on the stock are given by

$$\frac{dS(t)}{S(t)} + \frac{D(t)}{S(t)}dt = \mu(t)dt + \Sigma(t)dW_1(t).$$

Since the agents do not observe the true innovation $W_1(t)$, investor i perceives the dynamics of the return under his subjective probability measure $\mathbb{P}^i$ to be given by

$$\frac{dS(t)}{S(t)} + \frac{D(t)}{S(t)}dt = \mu_i(t)dt + \Sigma(t)d\nu_i(t),\; i = 1, 2,$$

where $\mu_i = \mathbb{E}_{\mathbb{P}^i}\left[\mu(t)|\mathcal{F}^{D,S}(t)\right]$.

3.3 The investor's optimization problems and demand functions

Each investor maximizes his expected utility from consumption by selecting the optimal consumption strategy $\{c_i(t)\}_{0\le t\le T}$ and investment strategy $\{\pi_i(t)\}_{0\le t\le T}$ such that his wealth, given by

$$dX_i(t) = (X_i(t)r(t) - c_i(t))dt + \pi_i(t)[(\mu(t) - r(t))dt + \Sigma(t)dW_1(t)],\; X(0) = x_i,$$

is nonnegative at all times, $X_i(t) \ge 0$, $0 \le t \le T$. That is, he solves the consumption Problem 2 as described in section 2.2. Defining as before

$$\theta(t) \equiv \Sigma(t)^{-1}(\mu(t) - r(t)),$$
$$\xi(t) \equiv \exp\left(-\int_0^t \theta(u)dW_1(u) - \frac{1}{2}\int_0^t \theta(u)^2 du - \int_0^t r(u)du\right),$$

the problem writes in the static form as

$$\max_c \mathbb{E}\left[\int_0^T u_i(c_i(t), t)dt\right]$$
$$s.t.\, \mathbb{E}\left[\int_0^T \xi(t)c_i(t)dt\right] \le x.$$

This optimization problem involves the unobservable terms $\xi(t)$. Detemple and Murthy (1994) have shown that the investment and consumption policies that are optimal for each investor in this incomplete information market solve as well the consumption problem in a market where the dynamics of the risky asset are given by

$$\frac{dS(t)}{S(t)} + \frac{D(t)}{S(t)}dt = \mu_i(t)dt + \Sigma(t)d\nu_i(t),\; i = 1, 2.$$

Investor i has complete information on the relevant parameters in this market, that is, he knows

$$\theta_i(t) \equiv \Sigma(t)^{-1}(\mu_i(t) - r(t)),$$
$$\xi_i(t) \equiv \exp\left(- \int_0^t \theta_i(u)' d\nu_i(u) - \frac{1}{2}\int_0^t \theta_i(u)'\theta_i(u) du - \int_0^t r(u) du \right).$$

The static form of the optimization problem under complete information can be written as

$$\max_c \mathbb{E}\left[\int_0^T u_i(c_i(t), t) dt\right]$$
$$s.t.\, \mathbb{E}\left[\int_0^T \xi_i(t) c_i(t) dt\right] \leq x.$$

The optimal consumption policy is the one derived in section 2.3 for investors with power utility,

$$\hat{c}_i(t) = \left(\frac{y_i \xi_i(t)}{\beta_i(t)}\right)^{-1/R}, \; i = 1, 2,$$

where the Lagrange multiplier is given by

$$y_i^{-1/R} = \frac{x_i}{\mathbb{E}\left[\int_0^T \xi_i(t)^\rho \beta_i(t)^{1/R} dt\right]}.$$

3.4 Equilibrium

For the derivation of the equilibrium processes for the interest rate and market price of risk, it is convenient to denote the disagreement between the investors, taking investor 1 as a reference, by

$$\triangle(t) \equiv \gamma_2(t) - \gamma_1(t).$$

As will be shown in Lemma 3, the dynamics of this measure of disagreement do not depend on the investor's wealth or preferences. The state price density $\xi_2(t)$ of investor 2 can be expressed in terms of $\xi_1(t)$,

$$\xi_2(t) = \xi_1(t)\eta(t) \tag{61}$$

where

$$\eta(t) \equiv \exp\left(- \int_0^t \Sigma(s)^{-1}\triangle(s) d\nu_1(s) + \frac{1}{2}\int_0^t (\Sigma(s)^{-1}\triangle(s))'(\Sigma(s)^{-1}\triangle(s)) ds \right). \tag{62}$$

$\eta(t)$ is determined by exogenous terms only. In equilibrium, aggregate endowment must equal total consumption at all times $0 \le t \le T$,

$$D(t) = \sum_{i=1}^{2} \hat{c}_i(t) = \left(\frac{y_1 \xi_1(t)}{\beta_1(t)}\right)^{-1/R} + \left(\frac{y_2 \xi_2(t)}{\beta_2(t)}\right)^{-1/R}. \tag{63}$$

Since the agents were assumed to have the same relative risk aversion R, we can solve the market clearing condition (63), using (61), for the state price density $\xi_1(t)$ of investor 1 as a function of the exogenously given endowment $D(t)$, $\eta(t)$ and $\beta_i(t)$, $i = 1, 2$, as well as the endogenous Lagrange multipliers $y_i, i = 1, 2$,

$$\xi_1(t)^{-1/R} = \frac{D(t)}{\left(\frac{y_1}{\beta_1(t)}\right)^{-1/R} + \left(\frac{y_2 \eta(t)}{\beta_2(t)}\right)^{-1/R}} = \frac{D(t)}{A(t, \eta(t); y_1, y_2)} \tag{64}$$

where $A(t, \eta(t); y_1, y_2) \equiv \left(\frac{y_1}{\beta_1(t)}\right)^{-1/R} + \left(\frac{y_2 \eta(t)}{\beta_2(t)}\right)^{-1/R}$.

Proposition 3. In equilibrium, the interest rate and the market price of risk are as follows:

$$\begin{aligned} r(t) = {} & \frac{R(A_0(t) + A_1(t)\gamma_1(t))}{D(t)} - \frac{1}{2}(1 + 1/R)\theta_1(t) \\ & - \frac{1}{RA(t, \eta(t); y_1, y_2)} \left(\frac{y_2}{\beta_2(t)}\right)^{-1/R} \eta(t)^{-1/R} B(t)^{-1/R} \triangle(t) \left(B(t)^{-1} \triangle(t)(R-1) + \theta_1(t)\right), \end{aligned} \tag{65}$$

$$\theta_1(t) = \frac{RB(t)}{D(t)} - A(t, \eta(t); y_1, y_2)^{-1} \left(\frac{y_2}{\beta_2(t)}\right)^{-1/R} \eta(t)^{-1/R} B(t)^{-1} \triangle(t), \tag{66}$$

$$\theta_2(t) = \theta_1(t) + \Sigma(t) B(t)^{-1} \triangle(t). \tag{67}$$

The Lagrange multipliers y_i, $i = 1, 2$ are solutions to the following system of equations:

$$y_1 = \mathbb{E}\left[\int_0^T A(v, \eta(v); y_1, y_2)^{R-1} D(v)^{1-R} \beta_1(v)^{1/R} dv\right]^R x_1^{-R}, \tag{68}$$

$$y_2 = \mathbb{E}\left[\int_0^T A(v, \eta(v); y_1, y_2)^{R-1} D(v)^{1-R} \beta_2(v)^{1/R} \eta(v)^{\rho} dv\right]^R x_2^{-R}. \tag{69}$$

Proof. See the Appendix. □

The expected rate of return $\mu_i(t)$ in equilibrium on the risky asset, as perceived by agent i, follows from the equilibrium interest rate and the market prices of risk, $\mu_i(t) = r(t) + \Sigma(t)\theta_i(t)$.

Proposition 3 is a straightforward generalization of the result obtained by

Detemple and Murthy (1994), who looked at logarithmic utility, to the case of power utility. The current paper differs in two aspects from Berrada (2002) who also considers power utility. Berrada (2002) studies in an economy with hetereogeneous beliefs of the same type as here the impact that incompleteness of information on the unobservable growth rate of aggregate consumption has on properties of asset prices (such as the equilibrium market price of risk and the volatility) under the objective probability measure, as they can be observed by an outside observer under the objective probability measure. He identifies conditions under which market price of risk and the volatility are different from the complete information case. He uses simplified dynamics of the aggregate endowment (he uses the specification $A_0(t, D(t)) \equiv 0$, $A_1(t, D(t)) = D(t)$ and $B(t, D(t)) = D(t)\lambda(t)$ for some process for the volatility $\lambda(t)$). Further, Berrada (2002) expresses the interest rate and market price of risk in equilibrium using the shares of consumption of the different agents. Yet, for the purpose of calculating the trading volume, it is more convenient to express the equilibrium quantities completely in terms of exogenous terms.

As an illustration of the results presented in Proposition 3, we consider the case of homogeneous beliefs, that is $\triangle(t) = 0$ and $V_0^{(1)} = V_0^{(2)}$. The equilibrium quantities become

$$
\begin{aligned}
r^h(t) &= R\frac{(A_0(t) + A_1(t)\gamma_1(t))}{D(t)} - \frac{1}{2}(1 + 1/R)\frac{RB(t)}{D(t)} \\
\theta_1^h(t) &= \frac{RB(t)}{D(t)} \\
\theta_2^h(t) &= \theta_1(t),
\end{aligned}
$$

where the superscript c denotes the complete information case. Denoting by $\mu^D(t) \equiv \frac{(A_0(t)+A_1(t)\gamma_1(t))}{D(t)}$ the expected growth rate of the consumption and by $\sigma^D(t) \equiv \frac{RB(t)}{D(t)}$ its volatility, we find for the interest rate

$$
\begin{aligned}
r^c(t) &= R\mu^D(t) - \frac{1}{2}PR\sigma^D(t) \\
\mu_1^h(t) - r(t) &= R\Sigma(t)\sigma^D(t) \\
\mu_2^h(t) - r(t) &= \mu_1(t) - r(t).
\end{aligned}
$$

where $P \equiv -\frac{u_i'''(c,t)}{u_i''(c,t)}c = (1 + R)$ denotes the prudence coefficient of the investor having power utility. Thus, when agents have homogeneous beliefs we find the consumption CAPM: The risk premium on the risk asset equals the covariance of the risky asset with the aggregate consumption multiplied by the aggregate risk aversion. Expressing the interest rate $r(t)$ and market price of risk $\theta(t)$, pertaining in the economy with heterogeneous beliefs, by the analogues in a homogeneous beliefs economy $r^h(t)$ and $\theta^h(t)$, we find

$$r(t) = r^c(t) + \frac{1}{2}(1 + 1/R) - A(t, \eta(t); y_1, y_2)^{-1} \left(\frac{y_2}{\beta_2(t)}\right)^{-1/R} \eta(t)^{-1/R} B(t)^{-1} \triangle(t)$$

$$\theta(t) = \theta^c(t) - A(t, \eta(t); y_1, y_2)^{-1} \left(\frac{y_2}{\beta_2(t)}\right)^{-1/R} \eta(t)^{-1/R} B(t)^{-1} \triangle(t).$$

Thus, the more investors disagree on the present expected growth rate, that is the larger $\triangle(t)$, the more the interest rate and the market price of risk will deviate from this homogenous beliefs economy. At a time where investors disagree, the whole history of their beliefs matters since the ratio of their state price densities, $\eta(t)$ as defined in (62), involves the disagreement at all times before time t, $\triangle(u)$, $0 \leq u \leq t$. Further, the level of the interest rate is also determined by the reference investor's estimate $\gamma_1(t)$ of the growth rate. Hence, whereas for the homogeneous beliefs model the only relevant exogenous quantity is the aggregate consumption, three additional quantities $\eta(t)$, $\triangle(t)$ and $\gamma_1(t)$ play a role in the heterogeneous beliefs economy.

3.5 Trading volume in equilibrium

Since the economy consists of only two investors, it suffices to analyze the trading volume of one investor, since his trading volume equals the aggregate trading volume. The disagreement among the investors creates trade since they continuously rebalance their portfolios.

As discussed above, the interest rate and the market price of risk depend in the model with heterogeneous beliefs not only on the aggregate consumption, but also on three further exogenous quantities, $\eta(t)$, $\triangle(t)$ and $\gamma_1(t)$. It will be shown below that these variables along with the aggregate consumption follow Ito processes. The interest rate (65) and the market price of risk (66) for the reference investor 1 are (differentiable) functions of the current value of these four quantities. Hence, the assumptions made for the analysis of section 2.4 are satisfied. We define a vector of four parameters,[7] $p = 4$,

$$Y(t) = \begin{pmatrix} Y_1(t) \\ Y_2(t) \\ Y_3(t) \\ Y_4(t) \end{pmatrix} = \begin{pmatrix} \gamma_1(t) \\ \triangle(t) \\ \eta(t) \\ D(t) \end{pmatrix}.$$

The dependence of the investment opportunity set on these parameters is as

[7] These four parameters have have a different interpretation than the state variables used by Merton (1973). Only the aggregate consumption is a state variable in the usual sense; the other quantities capture the characteristics of the beliefs of the agents in the economy.

follows. On one hand, the interest rate is given as a function (65) $r(t, Y(t)) = r(t, \gamma_1(t), \triangle(t), \eta(t)) = r(t, Y(t))$ (where (66) needs to be substituted for $\theta_1(t)$). On the other hand, this is completed by the market price of risk $\theta_1(t, Y(t)) = \theta_1(t, \gamma_1(t), \triangle(t), \eta(t)) = \theta_1(t, Y(t)$ given by (66). As for the dynamics of the parameters, $\gamma_1(t)$ solves the SDE (59), the dynamics of $\eta(t)$ follow by an application of Ito's lemma to the definition (62) of $\eta(t)$. The dynamics of $\triangle(t)$ are given in the following

Lemma 3. $\triangle(t)$ solves the following SDE:

$$d\triangle(t) = \left(a_1(t) - \left(\frac{A_1(t)}{B(t)}\right)^2 V_2(t)\right)\triangle(t)dt + \frac{A_1(t)}{B(t)}(V_2(t) - V_1(t))d\nu_1(t) \tag{70}$$

$$\triangle(0) = \gamma_2(0) - \gamma_1(0).$$

Proof. See the Appendix. □

The initial distribution of $\gamma(0)$, conditional an $D(0)$ was assumed to be normal. Since this distribution is completely characterized by the mean and the variance, to have common prior beliefs means that investors agree on the mean growth rate $\hat{\gamma}(0)$, that is $\triangle(0) = 0$, as well as the variance, that is $V_1(0) = V_2(0)$. They will then agree about the variance at all future times, $V_1(t) = V_2(t)$, since the variances solve the same ODE (60) with the same initial value. As assumed, this has the effect that $\triangle(t)$ in (70) remains at its initial value, that is zero. In the case where they agree only about the initial mean growth rate $\hat{\gamma}(0)$, that is $\triangle(0) = 0$, but not about the variance of the distribution of $\gamma(0)$, there will be a difference of beliefs in later periods since then $d\triangle(t) \neq 0$.

It follows from this Lemma, together with the dynamics of $\eta(t), \gamma_1(t)$ and $D(t)$ stated previously, that the vector of parameters $Y(t)$ solves the following SDE:

$$dY(t) = \mu^Y(t, Y(t))dt + \sigma^Y(t, Y(t))d\nu_1(t),$$

where

$$\mu^Y(t, Y(t)) = \begin{pmatrix} a_0(t) + a_1(t)Y_1(t) \\ \left(a_1(t) - \left(\frac{A_1(t)}{B(t)}\right)^2 V_2(t)\right) Y_2(t) \\ (\Sigma(t)^{-1}Y_2(t))^2 Y_3(t) \\ A_0(t) + A_1(t)Y_1(t) \end{pmatrix}, \quad \sigma^Y(t, Y(t)) = \begin{pmatrix} \frac{V_1(t)A_1(t)}{B_1(t)} \\ \frac{A_1(t)}{B(t)}(V_2(t) - V_1(t)) \\ -\Sigma(t)^{-1}Y_2(t)Y_3(t) \\ B(t) \end{pmatrix}.$$

We can now apply the results obtained for the optimal stock portfolio and trading volume in sections 2 and 2.4. The $d-$dimensional Brownian motion $W(t)$ and the filtration $\{\mathcal{F}(t)\}_{0 \leq t \leq T}$ from before are now replaced by the Brownian motion $\nu_1(t)$ and the filtration $\{\mathcal{F}^D(t)\}_{0 \leq t \leq T}$. The parameters $\theta(t)$ and $\xi(t)$

are replaced by the respective quantities observable to investor 1. The optimal stock portfolio (38) for investor 1 in the consumption problem becomes

$$\pi_1(t) = \frac{1}{R}\pi_1^{(mv)}(t)x_1(t) + \pi^{(h)}(t)w_1^{(h)}(t),\ j = i, c,$$

where

$$\begin{aligned}
\pi_1^{(mv)}(t) &\equiv \Sigma(t)^{-1}\theta_1(t), \\
x_1(t) &\equiv x\xi_1(t)^{-1}\,\mathbb{E}_t\left[\frac{\int_t^T \xi_1(v)^\rho \beta_1(v)^{1/R} dv}{\mathbb{E}\left[\int_t^T \xi_1(v)^\rho \beta_1(v)^{1/R} dv\right]}\right], \\
w_1^{(h)}(t) &\equiv -(1-1/R)x\,\mathbb{E}_t\left[\frac{\int_t^T \xi_1(v)^\rho \beta_1(v)^{1/R} h_t^{(1)}(v) dv}{\mathbb{E}\left[\int_t^T \xi_1(v)^\rho \beta_1(v)^{1/R} dv\right]}\right], \\
h_t^{(1)}(v)' &\equiv \int_t^v \partial_2 r(u)\Phi_t(u)du + \int_t^v (d\nu_1(u) + \theta_1(u)du)'\partial_2\theta_1(u)\Phi_t(u), v \geq t.
\end{aligned}$$

$\pi_1^{(mv)}(t)$ is the portfolio which is instantaneously mean-variance efficient given the expected rate of return $\mu_1(t)$ on the stock as perceived by investor 1. $x_1(t)$ is the current wealth of investor 1. $w_1^{(h)}(t)$ is his demand for the hedging portfolio $\pi^{(h)}(t)$, given his beliefs on the future. The interest rate and the market price of risk in $h_t^{(1)}(v)$ are the equilibrium interest rate (65) and market price of risk (66).

The investor's trading volume in the stock is thus given by (39),

$$\begin{aligned}
d\tilde{\pi}_1(t) = \frac{1}{R}&\left(x_1(t)d\pi_1^{(mv)}(t) + \pi_1^{(mv)}(t)(dx_1(t) - x_1(t)dG_1(t)) + d\pi_1^{(mv)}(t)dx_1(t)\right) \\
&+ \pi_1^{(h)}(t)\left(dw_1^{(h)}(t) - w_1^{(h)}(t)dG_1(t)\right) + d\pi_1^{(h)}(t)w_1^{(h)}(t) + d\pi_1^{(h)}(t)dw_1^{(h)}(t),\ j = i, c,
\end{aligned} \tag{71}$$

The impact of the heterogeneity of beliefs, as captured by the disagreement $\triangle(t)$, on the trading volume can not be derived from (71) directly. The numerical method proposed in section 2.5 will alow us to assess the effects of heterogeneity of beliefs.

3.6 Results from a Monte Carlo study

To analyze the properties of aggregate trading volume in the economy with incomplete information described above, a simulation study is conducted. To this end, particular processes need to be chosen for the aggregate endowment as well as for its growth rate.

The specification of the aggregate endowment (56) and the growth rate (57) admits a large class of processes, taking possibly nonlinearities in the endowment into account. Yet, a typical choice is (for other work using the same specification in the context of learning see e.g. Brennan and Xia (2001)) (linear) generalized Brownian motion for the aggregate consumption, providing a strictly positive process, and an Ornstein-Uhlenbeck process for the growth rate $\gamma(t)$, providing the feature of mean-reversion,

$$
\begin{aligned}
dD(t) &= \gamma(t)D(t)dt + \sigma_D D(t)dW_1(t), \\
d\gamma(t) &= \phi_\gamma(\bar{\gamma} - \gamma(t))dt + \sigma_\gamma dW_0(t).
\end{aligned}
$$

We use yearly data on the U.S. real per capita consumption from 1889 to 2003 from Shiller (2003), provided on Robert Shiller's web site[8]. The parameters of the Ornstein-Uhlenbeck process governing $\gamma(t)$ are estimated by Maximum Likelihood from the following discrete time version:

$$
\gamma_{t+1} = \gamma_t + \phi_\gamma(\bar{\gamma} - \gamma_t) + \sigma\epsilon_t,\ t = 2, 3, \ldots, N-1, \tag{72}
$$

where ϵ_t, $t = 2, 3, \ldots, N-1$, is iid normal with mean zero and variance 1. Table 1 shows the results of the estimation together with the other parameters used. The parameter estimates are highly significant and similar to the values found by Brennan and Xia (2001).

Table 4 (Table 6) shows the expected trading volume $\mathbb{E}\left[d\tilde{\pi}_0(0)\right]$ of the reference investor when the risk aversion in the economy equals 0.75 (3). Both components of the total trading volume are shown, one coming from trading related to the mean-variance efficient portfolio and the other from trading related to the hedging portfolio. The results are shown for a length of the investment time horizon of 1, 4, 7 and 10 years. Each line in Table 4 (Table 6) corresponds to a fixed pair of initial beliefs, $\mathbb{P}^{(i)}$, as represented by $\gamma_0^{(i)}$ and $V_0^{(i)}$, $i = 1, 2$. While the variance of γ_0 under the prior belief $\mathbb{P}^1$ of the reference agent is kept constant, $V_0^{(1)} = 0.05^2$, two values have been chosen for the corresponding variance under the prior belief $\mathbb{P}^2$ of the second investor, $V_0^{(2)} = 0.01^2$ and $V_0^{(2)} = 0.05^2$. Therefore, in the upper half of the table, the investors disagree on the variance, whereas in the lower half they agree. For both cases, the initial error that the reference investor makes with respect to the true moment of γ_0, $\delta \equiv \gamma_0^{(1)} - \hat{\gamma}_0$, is once chosen as zero and once as 0.1. The description of the investors' beliefs is completed by their means. Their disagreement on the mean of γ_0 ranges from -2.5% to $+2.5\%$. When $\triangle(0)$ is negative (positive), the second investor is more optimistic (pessimistic) than the reference investor. In the case where they both agree on the mean and the variance, that is, $\triangle(0) = \gamma_0^{(2)} - \gamma_0^{(1)} = 0$ and

[8] http://aida.econ.yale.edu/~shiller/data.htm

$V_0^{(2)} = 0.05^2 = V_0^{(1)}$, they have homogeneous beliefs. If, in addition, the reference investor makes no estimation error, $\delta = 0$, there is complete information. For each time horizon T, equations (68) and (69) are solved numerically for y_1 and y_2. The solution is then used to calculate approximations of the quantities determining the trading volume in an economy where the interest rate is given by (65) and the market price of risk by (66). 1000 paths have been simulated at a weekly time increments. As before, we lack error bounds for the statistical and numerical errors of the analysis. Yet, checks performed with 10'000 paths showed that the results are sufficiently stable with respect to the number of paths simulated. The investors' subjective discount factors have been chosen as $\beta_i(t) = 1$, $i = 1, 2, 0 \leq t \leq T$ to eliminate its impact.

As can be seen from Tables 4 and 6, there is a positive relationship between absolute size of expected trading volume and the absolute size of investors' disagreement on the mean as measured by $\triangle(0)$: A larger disagreement on the mean growth rate generally increases the absolute size of the expected trading volume. When they agree on the mean, $\triangle(0) = 0$, the expected trading volume tends to small in absolute size.

In the case of homogeneous beliefs, where investors agree both on the variance and the mean, $V_0^{(1)} = V_0^{(2)} = 0.05^2$ (the results for this case are shown in the lower parts of Tables 4 and 6) and $\triangle(0) = 0$, expected trading volume equals zero, subject to errors resulting from the numerical approximation. As mentioned above, in this case the model is reduced to the standard consumption CAPM without trading activity. The simulations show that in this case, the error that the reference investor makes has a negligible effect: Whether $\delta = 0$ or $\delta = 0.1$, expected trading volume is zero if investors agree, that is $\triangle(0) = 0$ and $V_0^{(1)} = V_0^{(2)}$. This shows that the driving forces of trading is disagreement, not the absolute error that an investor makes.

A further finding is that disagreement on the variance accentuates the effect that disagreement on the mean has on expected trading volume: Variation of the disagreement $\triangle(0)$ on the mean has a larger effect in the upper parts of Tables 4 and 6, where investors in addition disagree on the variance, than in the case where they agree on the variance, shown in the lower parts of the respective tables. This shows that moments of higher order have a significant effect on moments of the trading volume in the economy.

The length T of the investment time horizon tends to increase the absolute size of the expected trading volume. The effect is the strongest when T changes from 1 year to 4 years, its effect almost vanishes for longer horizons. Yet, this is not a simple consequence of the increased stock demand of investor's with longer time horizon, as observed for example in the economy of section 2.4: Different values of the length T of the time horizon correspond in the present economy to different pairs of Lagrange multipliers (y_1, y_2), implying different dynamics of the interest rate and the market price of risk. A change of the investment horizon is thus not *ceteris paribus* possible, since the dynamics of the interest rate and the market price of risk as endogeneous quantities will also change. In an economy where investors have a longer investment time horizon, they could demand even less stocks, due to different dynamic properties of the investment opportunity set. The tendency of the absolute size of the expected trading volume to increase is therefore produced by the heterogeneity of beliefs.

A higher degree of relative risk aversion R clearly decreases the expected trading volume. Also this is not a simple consequence of the result found in the partial equilibrium that more risk adverse investors invest a lower fraction of their wealth into the risky assets. In the current equilibrium model, the economies corresponding to an equilibrium for different degrees of relative risk aversion have different asset price dynamics that give rise to the drop in expected trading volume.

As for the components of expected trading volume, in general, a larger fraction stems from the trading related to the hedging component. For the more risk-averse investor with relative risk aversion of 3, almost all expected trading volume comes from the hedging component, given that he invests less money into the mean-variance efficient portfolio. Yet, the trading volume due to the mean-variance component of the investor's portfolio dominates total expected trading volume for investors with relative risk aversion of 0.75 and time horizon equal to one year when there is disagreement on the variance.

In Table 5 (Table 7), the volatility of trading volume is shown for a risk aversion of 0.75 (3).[9] The structure of the table is the same as for the expected trading volume. The driving force is also here the disagreement on the mean: Volatility of trading volume generally increases with the absolute size of the disagreement $\triangle(0)$, although this effect is only weak for a relative risk aversion of $R = 0.75$.

When investors disagree on the variance, the effect of additional disagreement on the mean on the volatility of trading volume is also stronger: The variation of the volatility with $\triangle(0)$ in Tables 5 and 7 is larger in the upper half, where there is disagreement on the variance, than in the lower half of the tables. Surprisingly, in the case of disagreement both on the mean and variance, this effect is only weak for the higher risk aversion of $R = 3$.

Risk aversion reduces volatility of trading volume: Generally, the number for the higher relative risk aversion of $R = 3$ shown in Table 7 are much smaller

[9]It would be interesting to analyze an economy in which agents have differing coefficients of relative risk aversion. But then the market clearing conditions cannot be solved for the equilibrium quantities.

than for a relative risk aversion of $R = 0.75$ shown in Table 5.

For less risk averse investors with a relative risk aversion of $R = 0.75$, volatility is the highest when the investment horizon is short. When investors' relative risk aversion is higher, the length T of the investment time horizon has mixed effects.

The results on trading volume reported in Tables 4 to 7 are expressed in multiples of the yearly US GDP. The actual yearly trading volume in USA amounted in 2003 to about twice the US GDP. [10]In order to compare the results from the simulation with actual trading volume, we have to fix a number of parameters. We focus here on a risk aversion of $R = 3$ which is commonly considered to be in a realistic range. It remains to fix values for the investment horizon T, the error δ of the reference investor, the disagreement $\triangle(0)$ and the variance $V_0^{(2)}$ as perceived by the second investor. From Table 6, we see that if investors disagree on the second moment, that is $V_0^{(2)} = 0.01^2 \neq 0.05^2 = V_0^{(1)}$, an expected trading volume of the order 1.5 to 2 results for a disagreement of 2.5% or 1%, depending on the size of the error δ of the reference investor. In the case where the investors agree on the variance, $V_0^{(2)} = 0.05^2$, trading volume is only sufficiently high for $\delta = 0.1$ and $\triangle(0) = \pm 2.5\%$. Thus, the model reproduces predictions on the expected trading volume close to actual numbers, when investors have differing beliefs on higher order moments. Otherwise, a relatively large difference of beliefs of $\pm 2.5\%$ is required to give rise to realistic levels of trading volume.

A shortcoming of the model is that it involves intractable direct as well as indirect effects of the variables and parameters on each other which makes it a difficult task to identify causal relationships. As an illustration, it is clear from (66) that the market price of risk $\theta_1(t)$ does not depend on the current perceived growth rate $\gamma_1(t)$. Yet, $\theta_1(t)$ depends at any point of time on the whole history of the differences of beliefs, as comprised in $\eta(t)$. Therefore, as an analysis of the demand for the hedging portfolios shows, the investor has a demand for the hedging portfolio against shifts in $Y_2(t) = \gamma_1(t)$ to hedge the market price of risk, $w_1^{(h,\theta_1)}(t) \neq 0$. A further illustration of these diverse cross relationships is the fact that the interest rate and the market price of risk at a given time t

[10]The average daily dollar value of trading at the NYSE was in 2003 38.5 billion USD (New York Stock Exchange, 2004); the analogue on the NASDAQ was in 2002 about 30 billion USD (NASDAQ, 2004); this gives a rough estimate of the yearly trading volume on these two exchanges of 15'000 billion USD. In contrast, the US GDP amounted in 2003 to 10'984 billion USD (Bureau of Economic Analysis, 2004).

depend on the whole history of the four variables in $Y(t)$:

$$r(t) = r(0) + \int_0^t \mu^r(s, \gamma_1(s), \triangle(s), \eta(s))ds + \int_0^t \sigma^r(s, \gamma_1(s), \triangle(s), \eta(s))d\nu_1(s)$$

$$\theta_1(t) = \theta_1(0) + \int_0^t \mu^{\theta_1}(s, \gamma_1(s), \triangle(s), \eta(s))ds + \int_0^t \sigma^{\theta_1}(s, \gamma_1(s), \triangle(s), \eta(s))d\nu_1(s),$$

where the coefficients of this linear SDE follow from an application of Ito's lemma to (65) and (66). Therefore, *a forteriori*, no simple relationships between today's value of the variables and the future investment opportunity set can be determined. Whereas in section 2.6 the stock demand and the trading volume could be explained by the sensitivity of the investment opportunity set with respect to shocks, such an analysis is not possible for the current model.

Summarizing, we find that disagreement on the mean increases the absolute size of the expected trading volume as well as its volatility. Additional disagreement on the variance makes this effect more pronounced. A longer investment horizon tends to induce higher expected trading volume. For investors with low relative risk aversion, volatility of trading volume is the highest for a short investment time horizon. In general, trading due to hedging reasons contributes a large part to the expected trading volume, while the hedging and mean-variance component are equally important for the volatility of trading volume. The model reproduces an expected trading volume close to actual trading volume when investors disagree both on the mean and variance. Otherwise, a relatively large difference of beliefs of $\pm 2.5\%$ is required to give rise to realistic levels of trading volume.

4 Conclusion

In this paper, the standard model of optimal investment and consumption in continuous time has been extended by the important aspect of trading volume. The fundamental shocks to the economy that affect the valuation of assets also drive the trading activity. It was shown that in the general case of an economy where state variables follow diffusion processes, volume is given by a diffusion process as well. It was found that trading has different motivations, coming from the demand for the mean-variance efficient portfolio and hedging portfolios in the sense of Merton (1973).

To characterize these different sources of trading volume quantitatively, a numerical analysis was performed in a particular economic setting, namely the one for which Detemple et al. (2003) analyzed the optimal stock portfolio where the interest rate and the market price of risk have nonlinear mean-reversion and the latter has hyperbolic elasticity of variance. It was found that the largest part of the expected volume stems from rebalancing the mean-variance portfolio and that therefore time horizon has little effect on expected volume. Both the expected size of trading volume related to the mean-variance portfolio as well as its volatility are lower for more risk adverse investors, reflecting their smaller investment into the mean-variance portfolio. Both the expected volume from the hedging portfolio as well as its volatility change sign at a risk aversion of 1. An investor with longer investment time horizon invests more funds into the hedging portfolio and this contributes to the expected trading volume. Whenever the investor has a relative risk aversion different from 1, hedging also contributes an important part to the volatility of trading volume.

Numerous research, as mentioned in the introduction, has analyzed the effect of asymmetric information and heterogeneous beliefs on trading volume in individual stocks. The general conclusion from this research is that differences in information and beliefs increase the trading volume. In this paper, we have found that in the case with symmetric information where agents have differing beliefs about the prospects of the economy, disagreement on the mean growth rate increase the absolute size of expected trading volume, and its impact is stronger when investors also disagree on the variance of the growth rate. A longer investment time horizon tends to increase absolute expected trading volume, having the largest effect when the horizon increases from 1 to 4 years. A large part of the expected trading volume comes from the hedging portfolio. Low risk aversion also implies high absolute expected trading volume. For investors with a low relative risk aversion, volatility of trading volume tends to be the highest for a short investment time horizon, while the effect of the investment time horizon is mixed for more risk averse investors. The model reproduces an expected trading volume close to actual trading volume when investors disagree both on the mean and variance. Otherwise, a relatively large difference of beliefs of $\pm 2.5\%$ is required to give rise to realistic levels of trading volume.

It would be interesting to extend the analysis presented in this paper to a non Gaussian framework. The use of, for example, Levy processes could yield a trading volume which is of bounded variation. Thus, it would be a better approximation of observed trading activity. Trading volume could then be accumulated over time and the properties of its intertemporal behavior could match observed volume data better. Further, it would be interesting to see how the results change for the economy with heterogeneous beliefs when there is an additional source of information, such as economic forecasts on the outlook of the economy and how the trading volume develops around such announcement dates.

References

Admati, A. R. (1988). A theory of intraday patterns: Volume and price variability. *Review of Financial Studies*, 1(1):3–40.

Ané, T. and Geman, H. (2000). Order flow, transaction clock, and normality of asset returns. *Journal of Finance*, 55(5):2259–2284.

Basak, S. (2000). A model of dynamic equilibrium asset pricing with heterogeneous beliefs and extraneous risk. *Journal of Economic Dynamics and Control*, 24:63–95.

Berrada, T. (2002). Incomplete information, heterogeneous beliefs, and the statistical properties of asset prices. Working paper.

Biais, B. and Bossaerts, P. (1998). Asset prices and trading volume in a beauty contest. *Review of Economic Studies*, 65(2):307–340.

Biais, B., Glosten, L., and Spatt, C. (2002). The microstructure of stock markets.

Blume, L., Easly, D., and O'Hara, M. (1994). Market statistics and technical analysis: The role of volume. *Journal of Finance*, 49(1):153–181.

Breeden, D. T. (1979). An intertemporal asset pricing model with stocahstic consumption and investment opportunities. *Journal of Financial Economics*, 7:265–296.

Brennan, M. J., Schwartz, E. S., and Lagnado, R. (1997). Strategic asset allocation. *Journal of Economic Dynamics and Control*, 21(8-9):1377–1403.

Brennan, M. J. and Xia, Y. (2001). Stock price volatility and equity premium. *Journla of Monetary Economics*, 47:249–283.

Buraschi, A. and Jiltsov, A. (2002). Uncertainty, volatility and option markets.

Buraschi, A. and Jiltsov, A. (2003). Option volume and differences in beliefs.

Bureau of Economic Analysis (2004). Current-dollar and real GDP. `http://www.bea.doc.gov/bea/dn/gdplev.xls`.

Copeland, T. E. (1976). A model of asset trading under the assumption of sequential information arrival. *Journal of Finance*, 31(4):1149–1168.

Copeland, T. E. (1977). A probability model of asset trading. *Journal of Financial and Quantitative Analysis*, 12:563–578.

Cox, J. C. and Huang, C.-f. (1989). Optimal consumption and portfolio policies when asset prices follow a diffusion process. *Journal of Economic Theory*, 49(1):33–83.

Cox, J. C. and Huang, C.-f. (1991). A variational problem arising in financial economics. *Journal of Mathematical Economics*, 20(5):465–487.

Detemple, J. and Murthy, S. (1994). Intertemporal asset pricing with heterogeneous beliefs. *Journal of Economic Theory*, 62:294–320.

Detemple, J. B., Garcia, R., and Rindisbacher, M. (2003). A monte carlo method for optimal portfolios. *Journal of Finance*, 58(1):401–446.

Foster, F. and Viswanathan, S. (1990). A theory of interday variations in volume, variance, and trading costs in securities markets. *Review of Financial Studies*, 3(4):593–624.

Foster, F. and Viswanathan, S. (1993). The effect of public information and competition on trading volume and price volatility. *Review of Financial Studies*, 6(1):23–56.

Gallant, R. A., Rossi, P. E., and Tauchen, G. (1992). Stock prices and volume. *Review of Financial Studies*, 5(2):199–242.

Gerety, M. S. and Mulherin, J. H. (1992). Trading halts and market activity: An analysis of volume at the open and the close. *Journal of Finance*, 47(5):1765–1784.

Grundy, B. D. and McNichols, M. (1989). Trade and the revelation of inforamtion through prices and direct disclosure. *Review of Financial Studies*, 2:495–526.

Harris, M. and Raviv, A. (1993). Differences of opinion makes a horse race. *Review of Financial Studies*, 6(3):473–506.

Jain, P. C. and Joh, G.-H. (1988). The dependence between hourly prices and trading volume. *Journal of Financial and Quantitative Analysis*, 23(3):269–283.

Jennings, R. H. and Barry, C. B. (1983). Information dissemination and portfolio choice. *Journal of Financial and Quantitative Analysis*, 36(1):1–19.

Jennings, R. H., Starks, L. T., and Fellingham, J. C. (1981). An equilibrium model of asset trading with sequential information arrival. *Journal of Finance*, 36(1):143–161.

Kandel, E. and Pearson, N. D. (1995). Differential interpretation of public signals and trade in speculative markets. *Journal of Political Economy*, 103(4):831–872.

Karatzas, I., Lehoczky, J., Sethi, S., and Shreve, S. (1986). Explicit solution of a general consumption/investment problem. *Mathematics of Operations Research*, 11(2):261–294.

Karatzas, I. and Shreve, S. E. (1998). *Methods of Mathematical Finance*, volume 39 of *Applications of Mathematics*. Springer, New York.

Karpoff, J. M. (1987). The relation between price changes and trading volume: A survey. *Journal of Financial and Quantitative Analysis*, 22(1):109–126.

Kloeden, P. E. and Platen, E. (1999). *Numerical Solution of Stochastic Differential Equations*. Number 23 in Applications of Mathematics. Springer, Berlin, 3rd edition.

Kyle, A. (1985). Continuous auctions and insider trading. *Econometrica*, 53:1315–1335.

Liptser, R. S. and Shiryaev, A. N. (2001). *Statistics of Random Processes*, volume 6 of *Applications of Mathematics: Stochastic Modelling and Applied Probability*. Springer, New York, 2 edition.

Merton, R. C. (1969). Lifetime portfolio selection under uncertainty: the continuous-time case. *Review of Economics and Statistics*, 51(3):247–257.

Merton, R. C. (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, 3(4):373–413.

Merton, R. C. (1973). An intertemporal capital asset pricing model. *Econometrica*, 41:867–887.

Merton, R. C. (1990). *Continuous-time finance*. Blackwell, Malden, Massachusetts.

Milgrom, P. and Stokey, N. (1982). Information, trade and common knowledge. *Journal of Economic Theory*, 26.

Morse, D. (1980). Asymmetrical information in securities markets and trading volume. *Journal of Financial and Quantitative Analysis*, 15(5):1129–1148.

NASDAQ (2004). Nasdaq market data: Nasdaq average daily dollar volume. `http://www.marketdata.nasdaq.com/asp/MaDollarVol.asp`.

New York Stock Exchange (2004). Nyse quick reference sheet. `http://www.nyse.com/marketinfo/1022221393893.html`.

Nualart, D. (1995). *The Malliavin Calculus and Related Topics*. Springer, New York.

Ocone, D. L. and Karatzas, I. (1991). A generalized Clark representation formula, with application to optimal portfolios. *Stochastics and Stochastics reports*, 34:187–220.

O'Hara, M. (1995). *Market Microstructure Theory*. Blackwell, Cambridge, Mass.

Radner, R. (1972). Existence of equilibrium of plans, prices and price expectations in a sequence of markets. *Econometrica*, 40:289–303.

Shalen, C. (1993). Volume, volatility, and the dispersion of beliefs. *Review of Financial Studies*, 6(2):405–434.

Shiller, R. J. (2003). From efficient markets theory to behavioral finance. *Journal of Economic Perspectives*, 17(1):83–104.

Ying, C. C. (1966). Stock market prices and volumes of sales. *Econometrica*, 34(3):676–685.

Zapatero, F. (1998). Effects of financial innovations on market volatility when beliefs are heterogeneous. *Journal of Economic Dynamics and Control*, 22:597–626.

A Proofs

Proof of Proposition 1 . To derive (16) from (8), we have to show that

$$(\Sigma(t)')^{-1}\bar{w}^{(h)}(t) = \pi^{(h)}(t)w^{(h)}(t) = (\Sigma(t)\Sigma(t)')^{-1}\Sigma(t)\sigma^Y(t)'w^{(h)}(t)$$

where $\bar{w}^{(h)}(t)$ was defined in (11) as

$$\bar{w}^{(h)}(t) = \begin{cases} -\mathbb{E}\left[\frac{\xi(T)}{\xi(t)}X_i(T)\left(1-\frac{1}{R_B(X_i(T))}\right)H_t(T)|\mathcal{F}(t)\right] & \text{for the investment problem,} \\ -\mathbb{E}\left[\int_t^T \frac{\xi(v)}{\xi(t)}\hat{c}(v)\left(1-\frac{1}{R_c(\hat{c}(v),v)}\right)H_t(v)dv|\mathcal{F}(t)\right] & \text{for the consumption problem} \end{cases}$$

$$H_t(v)' = \int_t^v \mathcal{D}_t\left(r(u)\right)du + \int_t^v (dW(u)+\theta(u)du)'\mathcal{D}_t\left(\theta(u)\right), v \geq t.$$

and $w^{(h)}(t)$ was defined in (18) as

$$w^{(h)}(t) = \begin{cases} -\mathbb{E}\left[\frac{\xi(T)}{\xi(t)}X_i(T)\left(1-\frac{1}{R_B(X_i(T))}\right)h_t(T)|\mathcal{F}(t)\right] & \text{for the investment problem,} \\ -\mathbb{E}\left[\int_t^T \frac{\xi(v)}{\xi(t)}\hat{c}(v)\left(1-\frac{1}{R_c(\hat{c}(v),v)}\right)h_t(v)dv|\mathcal{F}(t)\right] & \text{for the consumption problem,} \end{cases}$$

$$h_t(v)' \equiv \int_t^v \partial_2 r(u)\Phi_t(u)du + \int_t^v (dW(u)+\theta(u)du)'\partial_2\theta(u)\Phi_t(u), v \geq t.$$

The main step of the proof consists of transforming $\bar{w}^{(h)}(t)$ into $w^{(h)}(t)$. $\bar{w}^{(h)}(t)$ depends on $H_t(v)'$ and this contains the Malliavin derivatives of the interest rate and the market price of risk. Applying the chain rule of Malliavin calculus yields

$$\mathcal{D}_t\left(r(u)\right) = \partial_2 r(u)\mathcal{D}_t\left(Y(u)\right), \tag{73}$$

$$\mathcal{D}_t\left(\theta(u)\right) = \partial_2\theta(u)\mathcal{D}_t\left(Y(u)\right). \tag{74}$$

Noting that $Y(t)$ solves the system of linear SDE's given by (2), that is,

$$Y(t) = Y(0) + \int_0^t \mu^Y(u,Y(u))du + \int_0^t \sigma^Y(u,Y(u))dW(u),$$

we find that the Malliavin derivative of $Y(v)$ with respect to the Brownian motion k, $k = 1,2,\ldots,d$, equals

$$\mathcal{D}_{k,t}\left(Y(v)\right) = \int_t^v \partial_2\mu^Y(u,Y(u))\mathcal{D}_{k,t}\left(Y(u)\right)du + \sum_{j=1}^d \int_t^v \partial_2\sigma^Y_{\cdot j}(u,Y(u))\mathcal{D}_{k,t}\left(Y(u)\right)dW_j(u) + \sigma^Y_{\cdot k}(t,Y(t)).$$

This is a $p-$dimensional linear SDE. For notational convenience, we define

$$Z_{k,t}(v) \equiv \mathcal{D}_{k,t}\left(Y(v)\right),\ v \geq t$$
$$Z_{k,t}(t) = \sigma^Y_{\cdot k}(t, Y(t)).$$

The SDE writes as

$$dZ_{k,t}(u) = \partial_2 \mu^Y(u, Y(u)) Z_{k,t}(u) du + \sum_{j=1}^{d} \partial_2 \sigma^Y_{\cdot j}(u, Y(u)) Z_{k,t}(u) dW_j(u) \quad (75)$$
$$Z_{k,t}(t) = \sigma^Y_{\cdot k}(t, Y(t)). \quad (76)$$

The solution to this SDE is (see e.g. Kloeden and Platen (1999))

$$Z_{k,t}(u) = \Phi_t(u) Z_{k,t}(t) \quad (77)$$

where $\Phi_t(u)$ is the fundamental matrix satisfying

$$d\Phi_t(u) = \partial_2 \mu^Y(u, Y(u)) \Phi_t(u) du + \sum_{j=1}^{d} \partial_2 \sigma^Y_{\cdot j}(u, Y(u)) \Phi_t(u) dW_j(u),\ u \geq t \quad (78)$$
$$\Phi_t(t) = I. \quad (79)$$

The coefficients of the SDE (75) do not depend on k. Thus, the solution for any $Z_{k,t}(u)$ has the form (77), the product of the fundamental matrix with the initial value $Z_{k,t}(t)$. Collecting the d vectors $Z_{k,t}(u)$ in a $(d \times d)$ matrix $Z_t(u) = (Z_{1,t}(u), \ldots, Z_{d,t}(u))$, the solution writes as

$$Z_t(u) = \Phi_t(u) Z_t(t) = \Phi_t(u) \sigma^Y(t, Y(t)),$$

where the initial value (76) $Z_{k,t}(t) = \sigma^Y_{\cdot k}(t, Y(t))$ was taken into account. In terms of the Malliavin derivative, we obtained

$$\left(\mathcal{D}_{1,t}\left(Y(u)\right), \ldots, \mathcal{D}_{d,t}\left(Y(u)\right)\right) = \mathcal{D}_t\left(Y(u)\right) = \Phi_t(u) \sigma^Y(t, Y(t)),\ u \geq t. \quad (80)$$

It follows from substituting (48) into equation (46) and (47) that the Malliavin derivatives of the interest rate and the market price of risk can be replaced by

$$\mathcal{D}_t\left(r(u)\right) = \partial_2 r(u) \Phi_t(u) \sigma^Y(t), \quad (81)$$
$$\mathcal{D}_t\left(\theta(u)\right) = \partial_2 \theta(u) \Phi_t(u) \sigma^Y(t). \quad (82)$$

Substituting this into (81) and (82) yields

$$\mathcal{D}_t\left(r(u)\right) = \partial_2 r(u)\Phi_t(u)\sigma^Y(t, Y(t)),$$
$$\mathcal{D}_t\left(\theta(u)\right) = \partial_2 \theta(u)\Phi_t(u)\sigma^Y(t, Y(t)).$$

The term $H_t(v)$ becomes

$$\begin{aligned} H_t(v)' &= \int_t^v \partial_2 r(u)\Phi_t(u)\sigma^Y(t, Y(t))du + \int_t^v (dW(u) + \theta(u)du)'\partial_2 theta(u)\Phi_t(u)\sigma^Y(t, Y(t)) \\ &= \left(\int_t^v \partial_2 r(u)\Phi_t(u)du + \int_t^v (dW(u) + \theta(u)du)'\partial_2\theta(u)\Phi_t(u)\right)\sigma^Y(t, Y(t)) \\ &= h_t(v)'\sigma^Y(t, Y(t)). \end{aligned} \tag{83}$$

Thus, for the consumption problem,

$$\begin{aligned} (\Sigma(t)')^{-1}\bar{w}^{(h)}(t) &= -(\Sigma(t)')^{-1}\,\mathbb{E}\left[\int_t^T \frac{\xi(v)}{\xi(t)}\hat{c}(v)\left(1 - \frac{1}{R_c(\hat{c}(v), v)}\right)H_t(v)dv|\mathcal{F}(t)\right] \\ &= -(\Sigma(t)')^{-1}\,\mathbb{E}\left[\int_t^T \frac{\xi(v)}{\xi(t)}\hat{c}(v)\left(1 - \frac{1}{R_c(\hat{c}(v), v)}\right)\sigma^Y(t, Y(t))'h_t(v)dv|\mathcal{F}(t)\right] \\ &= -(\Sigma(t)')^{-1}\sigma^Y(t, Y(t))'\,\mathbb{E}\left[\int_t^T \frac{\xi(v)}{\xi(t)}\hat{c}(v)\left(1 - \frac{1}{R_c(\hat{c}(v), v)}\right)h_t(v)dv|\mathcal{F}(t)\right] \\ &= -(\Sigma(t)')^{-1}(\Sigma(t))^{-1}(\Sigma(t))\sigma^Y(t, Y(t))'\,\mathbb{E}\left[\int_t^T \frac{\xi(v)}{\xi(t)}\hat{c}(v)\left(1 - \frac{1}{R_c(\hat{c}(v), v)}\right)h_t(v)dv|\mathcal{F}(t)\right] \\ &= (\Sigma(t)\Sigma(t)')^{-1}\Sigma(t)\sigma^Y(t, Y(t))'w^{(h)}(t), \end{aligned}$$

where the third equality is due to our assumption that $\sigma^Y(t, Y(t))$ as a function of the state variables is measurable with respect to the filtration $\{\mathcal{F}(t)\}_{0\leq t\leq T}$ generated by the Brownian motion. Analogous equations hold for the investment problem. □

Proof of Proposition 2 . We first derive the dynamics of the optimal stock portfolio, representing the total change of the the stock positions' value. This equals the sum of the investor's gains on his portfolio's stock positions plus his trading volume in the stocks. If all components of the portfolio (25) are semi-martingales, we can derive $d\pi(t)$ by applying Ito's lemma. As a result of the assumed Markovian structure of the economy where the relevant exogenous quantities are function of the current state of the economy only, they are all Ito processes, according to the following

Lemma 4. The market price of risk $\theta(t)$ has the dynamics

$$d\theta(t) = \mu^{\theta}(t, Y(t))dt + \sigma^{\theta}(t, Y(t))dW(t), \tag{84}$$

the k-th column of the inverse of the diffusion matrix $\bar{\Sigma}(t) \equiv (\Sigma')^{-1}$,

$$d\bar{\Sigma}_{\cdot k}(t) = \mu_k^{\bar{\Sigma}}(t, Y(t))dt + \sigma_k^{\bar{\Sigma}}(t, Y(t))dW(t),\ k = 1, 2, \ldots, d, \tag{85}$$

and the dynamics of the (i, j) element of the diffusion matrix $\sigma^Y(t)$ of the state variables is given by

$$d\sigma_{i,j}^Y(t) = \mu_{i,j}^{\sigma^Y}(t, Y(t))dt + \sigma_{i,j}^{\sigma^Y}(t, Y(t))dW(t),\ i = 1, \ldots, p,\ j = 1, \ldots, d. \tag{86}$$

Proof of Lemma 4 . Since $\theta(t, Y(t))$ is a twice continuously differentiable function of time and the state variables, application of Ito's lemma yields (84) where

$$\mu^{\theta}(t, Y(t)) \equiv \partial_1\theta(t, Y(t)) + \partial_2\theta(t, Y(t))\mu^Y(t, Y(t)) + \frac{1}{2}\partial_{2,2}\theta(t, Y(t))\sigma^Y(t, Y(t))'\sigma^Y(t, Y(t)) \tag{87}$$

and

$$\sigma^{\theta}(t, Y(t)) \equiv \partial_2\theta\sigma^Y(t, Y(t)). \tag{88}$$

Also, $\Sigma_t = \Sigma(t, Y(t))$ is a $(d \times d)$ matrix of twice continuously differentiable functions $S_{ij}(t, Y(t))$ of time and the state variables. The i, j element of $\bar{\Sigma} = (\Sigma')^{-1}$,

$$s_{i,j}(t, Y(t)) = \frac{|(\Sigma(t, Y(t)))_{i,j}|}{|\Sigma(t, Y(t))|},$$

where $(\Sigma(t, Y(t)))_{i,j}$ denotes the ji-th cofactor of $\Sigma(t, Y)$, is the ratio of the elements of $\Sigma(t, Y(t))$ and thus again a continuously differentiable function. The dynamics of the k-th column $\bar{\Sigma}_{\cdot k}$ therefore are

$$d\bar{\Sigma}_{\cdot k}(t) = \mu_k^{\bar{\Sigma}}(t, Y(t))dt + \sigma_k^{\bar{\Sigma}}(t, Y(t))dW(t),\ k = 1, 2, \ldots, d, \tag{89}$$

where the $(d \times d)$ matrix $\mu^{\bar{\Sigma}}(t, Y(t)) = (\mu_1^{\bar{\Sigma}}(t, Y(t), \ldots, \mu_d^{\bar{\Sigma}}(t, Y(t))$ and the d $(d \times d)$ matrices $\sigma_k^{\bar{\Sigma}}(t, Y(t))$ are determined element by element from applying Ito's lemma to $s_{i,j}$.

Analogously, one arrives at the dynamics (86) of the (i, j) element of $\sigma^Y(t, Y(t))$, where

$$\mu_{i,j}^{\sigma^Y}(t,Y(t)) \equiv \partial_1\sigma_{i,j}^Y(t,Y(t)) + \partial_2\sigma_{i,j}^Y(t,Y(t))\mu^Y(t,Y(t)) + \frac{1}{2}\sum_{k=1}^{p}\sum_{l=1}^{p}\frac{\partial_2\sigma_{i,j}^Y(t,Y(t))}{\partial Y_k \partial Y_l}\sigma_{k\cdot}^Y(t,Y(t))\sigma_{l\cdot}^Y(t,Y(t)') \tag{90}$$

$$\sigma^Y(t,Y(t))_k \equiv \partial_2\sigma_{i,j}^Y(t,Y(t))\sigma^Y(t,Y(t)). \tag{91}$$

□

Continuing the proof Proposition 2, we apply Ito's lemma to the optimal portfolio (25) for an investor with constant relative risk aversion. This yields the following dynamics of the portfolio value:

$$\begin{aligned} d\pi(t) = &\frac{1}{R}(X_j(t)d\pi^{(mv)}(t) + \pi^{(mv)}(t)dX_j(t) + d\pi^{(mv)}(t)dX_j(t)) \\ &+ \pi^{(h)}(t)dw_j^{(h)}(t) + d\pi^{(h)}(t)w_j^{(h)}(t) + d\pi^{(h)}(t)dw_j^{(h)}(t),\ j = i, c. \end{aligned} \tag{92}$$

The trading volume is this quantity net of the capital gains, which are given by

$$\pi_k(t)dG_k(t) = \left(\frac{1}{R}X_j(t)\pi_k^{(mv)}(t) + \pi_k^{(h)}(t)w^{(h)}(t)\right)dG_k(t), \tag{93}$$

thus, subtracting these capital gains from (92) yields (39).

It remains to derive the dynamics of the wealth (40) and of the demand (41) for the hedging portfolios. For the investment problem, wealth is equal according to (26)

$$X_i(t) = \xi(t)^{-1}\,\mathbb{E}_t\left[\xi(T)I_B(y\xi(T))\right].$$

To derive the dynamics of $M(t) \equiv \mathbb{E}_t\left[\xi(T)I_B(y\xi(T))\right]$, we apply the Clark Ocone formula

$$\mathbb{E}\left[F|\mathcal{F}(t)\right] = \mathbb{E}\left[F\right] + \int_0^t \mathbb{E}_v\left[\mathcal{D}_v\left(F\right)\right]' dW(v)$$

to $F = \xi(T)I_B(y\xi(T))$, and obtain

$$\begin{aligned} d(M(t)) &= \mathbb{E}_t\left[\mathcal{D}_t\left(\xi(T)I_B(y\xi(T))\right)\right]' dW(t) \\ &= \mathbb{E}_t\left[\mathcal{D}_t\left(\xi(T)\right)\left(I_B(y\xi(T)) + y\xi(T)I_B'(y\xi(T))\right)\right]' dW(t) \end{aligned}$$

Applying the chain rule of Malliavin calculus yields (Detemple et al., 2003, p. 438) $\mathcal{D}_t\left(\xi(T)\right) = -\xi(T)(\theta(t)' + H_t(T))$. Further, $I_B(y\xi(T)) + y\xi(T)I_B'(y\xi(T)) = X_i(T) + B'(X_i(T))/B''(X_i(T)) = X_i(T)(1-1/R)$ since $I'(x) = 1/B''(x)$, $x > 0$.

This yields

$$\begin{aligned}
dM(t) &= -\mathbb{E}_t\left[\xi(T)(\theta(t)' + H_t(T))X_i(T)(1-1/R)\right]' dW(t)\\
&= -(1-1/R)(\theta(t)'\,\mathbb{E}_t\left[\xi(T)X_i(T)\right] + \mathbb{E}_t\left[\xi(T)X_i(T)H_t(T)\right])'dW(t)\\
&= -(1-1/R)(\theta(t)\xi(t)X_i(t) + \mathbb{E}_t\left[\xi(T)X_i(T)H_t(T)\right])'dW(t)
\end{aligned}$$

since $\theta(t)$ is $\mathcal{F}(t)$ measurable and $\mathbb{E}_t\left[\xi(T)X_i(T)\right] = \xi(t)X_i(t)$. Applying Ito's lemma to $X_i(t)$, we obtain

$$\begin{aligned}
dX_i(t) &= -\xi(t)^{-2}M(t)d\xi(t) + \xi(t)^{-1}dM(t) + \frac{1}{2}(2\xi(t)^{-3}M(t)d\langle\xi(t)\rangle - 2\xi(t)^{-2}d\langle\xi(t), M(t)\rangle)\\
&= \xi(t)^{-2}M(t)\xi(t)(r(t)dt + \theta(t)'dW(t)) - \xi(t)^{-1}((1-1/R)(\theta(t)\xi(t)X_i(t) + \mathbb{E}_t\left[\xi(T)X_i(T)H_t(T)\right])'dW\\
&+ \xi(t)^{-3}M(t)\xi(t)^2\theta(t)'\theta(t)dt - \xi(t)\theta(t)'(1-1/R)(\theta(t)\xi(t)X_i(t) + \mathbb{E}_t\left[\xi(T)X_i(T)H_t(T)\right])dt\\
&= X_i(t)(r(t)dt + \theta(t)'dW(t)) - ((1-1/R)\theta(t)'X_i(t) + (1-1/R)\xi(t)^{-1}\,\mathbb{E}_t\left[\xi(T)X_i(T)H_t(T)\right])'dW(t))\\
&+ X_i(t)\theta(t)'\theta(t)dt - (1-1/R)(\theta(t)'\theta(t)X_i(t)dt - (1-1/R)(\theta(t)'\xi(t)^{-1}\,\mathbb{E}_t\left[\xi(T)X_i(T)H_t(T)\right]dt\\
&= (X_i(t)r(t) + \frac{1}{R}X_i(t)\theta(t)'\theta(t) - (1-1/R)\xi(t)^{-1}\theta(t)'\,\mathbb{E}_t\left[\xi(T)X_i(T)H_t(T)\right])dt\\
&+ (\frac{1}{R}X_i(t)\theta(t)' - \xi(t)^{-1}(1-1/R)\,\mathbb{E}_t\left[\xi(T)X_i(T)H_t(T)\right]')dW(t).
\end{aligned}$$

where we used the fact that $X_i(t) = \xi(t)^{-1}M(t)$. Further, we notice that from (83) follows

$$\begin{aligned}
-(1-1/R)\xi(t)^{-1}\,\mathbb{E}_t\left[\xi(T)X_i(T)H_t(T)\right] &= -(1-1/R)\xi(t)^{-1}\,\mathbb{E}_t\left[\xi(T)X_i(T)\sigma^Y(t)'h_t(T)\right]\\
&= -(1-1/R)\xi(t)^{-1}\sigma^Y(t)'\,\mathbb{E}_t\left[\xi(T)X_i(T)h_t(T)\right]\\
&= \sigma^Y(t)w^{(h)}(t)
\end{aligned}$$

according to definition (18) of $w^{(h)}(t)$. Hence, we find (40)

$$dX_i(t) = \left[X_i(t)r(t) + \frac{1}{R}X_i(t)\theta(t)'\theta(t) + \theta(t)'\sigma^Y(t)w^{(h)}(t)\right]dt + \left[\frac{1}{R}X_i(t)\theta(t) + \sigma^Y(t)'w^{(h)}(t)\right]' dW(t).$$

The dynamics of $X_c(t)$ follow along the same lines.

To show (41), we change notation and we define now $M_{i,k}(t) \equiv \mathbb{E}_t\left[\xi(T)I_B(y\xi(T))h_{k,t}(T)\right]$ and $M_{c,k}(t) \equiv \mathbb{E}_t\left[\int_t^T \xi(v)\hat{c}(v)h_{k,t}(v)dv\right]$. Using this notation, the demand for the hedging portfolio writes as

$$w^{(h)}_{j,k}(t) = -\rho\xi(t)^{-1}M_{j,k}(t),\ j = i, c. \tag{94}$$

We first show that the dynamics of $M_{j,k}(t)$ are given by

$$dM_{j,k}(t) = \tilde{M}_{j,k}(t)'dW(t), \tag{95}$$

where $\tilde{M}_{j,k}(t)$, $j = i, c$ is defined in (42) and (43). We show (95) for the investment problem, it follows along the same lines for the consumption problem. As above, we apply the Clark Ocone formula

$$\mathbb{E}\left[F|\mathcal{F}(t)\right] = \mathbb{E}\left[F\right] + \int_0^t \mathbb{E}_v\left[\mathcal{D}_v\left(F\right)\right]' dW(v)$$

to $F = \xi(T)I_B(y\xi(T))h_{k,t}(T)$, and obtain

$$\begin{aligned}
dM_{i,k}(t) &= \mathbb{E}_t\left[\mathcal{D}_t\left(\xi(T)I_B(y\xi(T))h_{k,t}(T)\right)\right]' dW(t) \\
&= \mathbb{E}_t\left[\mathcal{D}_t\left(\xi(T)\right)\left(I_B(y\xi(T))h_{k,t}(T) + \xi(T)h_{k,t}(T)I_B'(y\xi(T))y\right) + \xi(T)I_B(y\xi(T))\mathcal{D}_t\left(h_{k,t}(T)\right)\right]' dW(t) \\
&= \mathbb{E}_t\left[h_{k,t}(T)\mathcal{D}_t\left(\xi(T)\right)\left(X_i(T) + \xi(T)I_B'(y\xi(T))y\right) + \xi(T)X_i(T)\mathcal{D}_t\left(h_{k,t}(T)\right)\right]' dW(t) \\
&= \mathbb{E}_t\left[-h_{k,t}(T)\xi(T)(\theta(t) + H_t(T))(X_i(T) + y\xi(T)I_B'(y\xi(T))) + \xi(T)X_i(T)\mathcal{D}_t\left(h_{k,t}(T)\right)\right]' dW(t)
\end{aligned}$$

We use again the result that $X(T) + y\xi(T)I_B'(y\xi(T)) = X_i(T)(1 - 1/R)$ as well as equality (83) and obtain

$$\begin{aligned}
dM_{i,k}(t) &= \mathbb{E}_t\left[-(1 - 1/R)h_{k,t}(T)X_i(T)\xi(T)(\theta(t) + H_t(T)) + \xi(T)X_i(T)\mathcal{D}_t\left(h_{k,t}(T)\right)\right]' dW(t) \\
&= \mathbb{E}_t\left[\xi(T)X_i(T)[-(1 - 1/R)(\theta(t) + \sigma^Y(t)'h_t(T))h_{k,t}(T) + \mathcal{D}_t\left(h_{k,t}(T)\right)]\right]' dW(t) \\
&= \tilde{M}_{i,k}(t)' dW(t).
\end{aligned}$$

To derive (41), we apply Ito's lemma to (94),

$$\begin{aligned}
dw_{j,k}^{(h)}(t) &= \rho\xi(t)^{-2}M_{j,k}(t)d\xi(t) - \rho\xi(t)^{-1}dM_{j,k}(t) + \rho\xi(t)^{-2}d\langle\xi(t), M_{j,k}(t)\rangle - \rho\xi(t)^{-3}M_{,kj}(t)d\langle\xi(t)\rangle \\
&= -\rho\xi(t)^{-1}M_{j,k}(t)(r(t)dt + \theta(t)'dW(t)) - \rho\xi(t)^{-1}\tilde{M}_{j,k}(t)'dW(t) - \rho\xi(t)^{-1}\theta(t)'\tilde{M}_{j,k}(t)dt \\
&- \rho\xi(t)^{-1}M_{j,k}(t)\theta(t)'\theta(t)dt \\
&= w_{j,k}^{(h)}(t)(r(t)dt + \theta(t)'dW(t)) - \rho\xi(t)^{-1}\tilde{M}_{j,k}(t)'(dW(t) - \theta(t)dt) + w_{j,k}^{(h)}(t)\theta(t)'\theta(t)dt
\end{aligned}$$

□

Proof of Lemma 1 .

$$\begin{aligned}
\mathbb{E}\left[d\tilde{\pi}(t)\right] &= \frac{1}{R}\left(X_j(t)\,\mathbb{E}\left[d\pi^{(mv)}(t)\right] + \pi^{(mv)}(t)\,\mathbb{E}\left[(dX_j(t) - X_j(t)dG(t))\right]\right) \\
&+ \pi^{(h)}(t)\,\mathbb{E}\left[(dw_j^{(h)}(t) - w_j^{(h)}(t)dG(t))\right] + w_j^{(h)}(t)'\,\mathbb{E}\left[d\pi^{(h)}(t)'\right] \\
&= \frac{1}{R}\left(X_j(t)\,\mathbb{E}\left[d\pi^{(mv)}(t)\right] - \pi^{(mv)}(t)X_j(t)\mu(t)dt\right) \\
&- \pi^{(h)}(t)w_j^{(h)}(t)\mu(t)dt + w_j^{(h)}(t)'\,\mathbb{E}\left[d\pi^{(h)}(t)'\right] \\
&= \frac{1}{R}X_j(t)\,\mathbb{E}\left[d\pi^{(mv)}(t)\right] + w_j^{(h)}(t)'\,\mathbb{E}\left[d\pi^{(h)}(t)'\right] - \pi(t)\mu(t)dt,\ j = i, c.
\end{aligned}$$

The diffusion term follows straight from the SDE (39) governing the trading volume $d\tilde{\pi}_k(t)$. □

Lemma 5. The Malliavin derivative of the i$-th$ component of $h_t(u)$, denoted by $h_{i,t}(u)$, $i = 1, 2, \ldots, p$, with respect to the Brownian motion $W_k(t)$, $k = 1, 2, \ldots, d$, solves the SDE

$$
\begin{aligned}
d\mathcal{D}_{j,t}\left(h_{i,t}(u)\right) = \sum_{k=1}^{p} \Bigg\{ & \sum_{l=1}^{p} \frac{\partial^2 r(u)}{\partial Y_k \partial Y_l} \left(\sum_{r=1}^{p} \Phi_{l,r,t}(u) \sigma^Y_{r,j}(t) \right) \Phi_{k,i,t}(u) du \\
& + \frac{\partial r(u)}{\partial Y_k} \mathcal{D}_{j,t}\left(\Phi_{k,i,t}(u)\right) du \\
& + \sum_{m=1}^{p} \Bigg[\left(\sum_{l=1}^{p} \frac{\partial^2 \theta_k(u)}{\partial Y_m \partial Y_l} \left(\sum_{r=1}^{p} \Phi_{l,r,t}(u) \sigma^Y_{r,j}(t) \right) \Phi_{m,i,t}(u) (dW_k(u) + \theta_k(u) du) \right) \\
& + \frac{\partial \theta_k(u)}{\partial Y_m} \mathcal{D}_{j,t}\left(\Phi_{m,i,t}(u)\right) (dW_k(u) + \theta_k(u) du) \\
& + \frac{\partial \theta_k(u)}{\partial Y_m} \Phi_{m,i,t}(u) \sum_{l=1}^{p} \frac{\partial \theta_k(u)}{\partial Y_l(u)} \left(\sum_{r=1}^{p} \Phi_{l,r,t}(u) \sigma^Y_{r,j}(t) \right) du \Bigg] \Bigg\}
\end{aligned}
\tag{96}
$$

$$
\lim_{u \to t} \mathcal{D}_{j,t}\left(h_{i,t}(u)\right) = \frac{\partial \theta_j(u)}{\partial Y_j} \tag{97}
$$

The Malliavin derivative of the (i,j) component of $\Phi_t(u)$, denoted by $\Phi_{i,j,t}(u)$, $i,j = 1, 2, \ldots, p$, with respect to the Brownian motion $W_n(t)$, $n = 1, 2, \ldots, d$, solves the SDE

$$
\begin{aligned}
d\mathcal{D}_{n,t}\left(\Phi_{i,j,t}(u)\right) = & \sum_{m=1}^{p} \left(\sum_{l=1}^{p} \frac{\partial^2 \mu_Y(u)}{\partial Y_m \partial Y_l} \left(\sum_{r=1}^{p} \Phi_{l,r,t}(u) \sigma^Y_{r,n}(t) \right) \Phi_{m,j,t}(u) + \frac{\partial \mu^Y_i(u)}{\partial Y_m} \mathcal{D}_{n,t}\left(\Phi_{m,j,t}(u)\right) \right) du \\
& + \sum_{k} \Bigg[\sum_{m=1}^{p} \sum_{l=1}^{p} \frac{\partial^2 \sigma^Y_{i,k}(u)}{\partial Y_m \partial Y_l} \left(\sum_{r=1}^{p} \Phi_{l,r,t}(u) \sigma^Y_{r,n}(t) \right) \Phi_{m,j,t}(u) \\
& + \frac{\partial \sigma^Y_{ik}(u)}{Y_m} \mathcal{D}_{n,t}\left(\Phi_{m,j,t}(u)\right) \Bigg] dW_k(u)
\end{aligned}
\tag{98}
$$

$$
\lim_{u \to t} \mathcal{D}_{n,t}\left(\Phi_{i,j,t}(u)\right) = \sum_{m=1}^{p} \frac{\partial \sigma^Y_{i,n}(u)}{\partial Y_m} \Phi_{m,j,t}(u). \tag{99}
$$

Proof of Lemma 5 . From (19), (20) and (21), $h_t(v)$ is defined as

$$
h_t(v)' = h^r_t(v)' + h^\theta_t(v)' = \int_t^v \partial_2 r(u) \Phi_t(u) du + \int_t^v (dW(u) + \theta(u) du)' \partial_2 \theta(u) \Phi_t(u)
$$

The i-th component of $h_t^r(v)$, $i=1,\dots,p$, writes as

$$h_{i,t}^r(v) = \int_t^v \sum_{k=1}^p \frac{\partial r(u,Y(u))}{\partial Y_k} \Phi_{ki,t}(u) du$$

The Malliavin derivative of $h_{i,t}^r(v)$ with respect to the j-th Brownian motion equals, due to the chain rule of Malliavin calculus,

$$\mathcal{D}_{j,t}\left(h_{i,t}^r(v)\right) = \int_t^v \sum_{k=1}^p \left(\sum_{l=1}^p \frac{\partial^2 r(u,Y(u))}{\partial Y_k \partial Y_l} \Phi_{ki,t}(u) \mathcal{D}_{j,t}\left(Y_l(u)\right) + \frac{\partial r(u,Y(u))}{\partial Y_k} \mathcal{D}_{j,t}\left(\Phi_{ki,t}(u)\right) \right) du.$$

The i-th component of $h_t^\theta(v)$, $i=1,\dots,p$, writes as

$$\begin{aligned} h_{i,t}^\theta(v) &= \int_t^v \sum_{k=1}^d (dW_k(u) + \theta_k(u)du) \left(\sum_{m=1}^p \frac{\partial \theta_k(u,Y(u))}{\partial Y_m} \Phi_{mi,t}(u) \right) \\ &= \int_t^v \sum_{k=1}^d \left(\sum_{m=1}^p \frac{\partial \theta_k(u,Y(u))}{\partial Y_m} \Phi_{mi,t}(u) \right) dW_k(u) + \int_t^v \sum_{k=1}^d \left(\sum_{m=1}^p \frac{\partial \theta_k(u,Y(u))}{\partial Y_m} \Phi_{mi,t}(u) \right) \theta_k(u) du \end{aligned}$$

The Malliavin derivative of the first term equals

$$\begin{aligned} &\sum_{m=1}^p \frac{\partial \theta_j(t,Y(t))}{\partial Y_m} \Phi_{mi,t}(t) + \int_t^v \sum_{k=1}^d \mathcal{D}_{j,t} \left(\sum_{m=1}^p \frac{\partial \theta_k(u,Y(u))}{\partial Y_m} \Phi_{mi,t}(u) \right) dW_k(u) \\ &= \sum_{m=1}^p \frac{\partial \theta_j(t,Y(t))}{\partial Y_m} \Phi_{mi,t}(t) + \int_t^v \sum_{k=1}^d \left(\sum_{m=1}^p \sum_{l=1}^p \frac{\partial^2 \theta_k(u,Y(u))}{\partial Y_m \partial Y_l} \Phi_{mi,t}(u) \mathcal{D}_{j,t}\left(Y_l(u)\right) + \frac{\partial \theta_k(u,Y(u))}{\partial Y_m} \mathcal{D}_{j,t}\left(\Phi_{mi,t}\right.\right. \end{aligned}$$

As for the second term,

$$\begin{aligned} &\int_t^v \sum_{k=1}^d \mathcal{D}_{j,t} \left(\sum_{m=1}^p \theta_k(u) \frac{\partial \theta_k(u,Y(u))}{\partial Y_m} \Phi_{mi,t}(u) du \right) \\ &= \int_t^v \sum_{k=1}^d \sum_{m=1}^p \left[\sum_{l=1}^p \frac{\partial \theta_k(u,Y(u))}{\partial Y_l} \frac{\partial \theta_k(u,Y(u))}{\partial Y_m} \Phi_{mi,t}(u) \mathcal{D}_{j,t}\left(Y_l(u)\right) du \right. \\ &\left. + \sum_{l=1}^p \frac{\partial^2 \theta_k(u,Y(u))}{\partial Y_m \partial Y_l} \Phi_{mi,t}(u) \mathcal{D}_{j,t}\left(Y_l(u)\right) \theta_k(u) du + \frac{\partial \theta_k(u,Y(u))}{\partial Y_m} \mathcal{D}_{j,t}\left(\Phi_{mi,t}(u)\right) \theta_k(u) du \right] \end{aligned}$$

When equality (48),

$$\mathcal{D}_{j,t}\left(Y_l(v)\right) = \sum_{r=1}^p (\Phi_{lr,t} \sigma_{rj}^Y),$$

is substituted for $\mathcal{D}_{j,t}\left(Y_l(v)\right)$, the claim follows. The boundary condition comes from the first term of $h^{\theta}_{i,t}(v)$, in which

$$\sum_{m=1}^{p} \frac{\partial \theta_j(u)}{\partial Y_m} \Phi_{m,i,t}(u)$$

is constant. Thus,

$$\lim_{u \to t} \mathcal{D}_{j,t}\left(h_{i,t}(u)\right) = \sum_{m=1}^{p} \frac{\partial \theta_j(t)}{\partial Y_m} \Phi_{m,i,t}(t) = \frac{\partial \theta_j(t)}{\partial Y_j},$$

since $\Phi_t(t) = I$.

Similarly, according to (22),

$$\Phi_t(v) = I + \int_t^v \partial_2 \mu^Y(u) \Phi_t(u) du + \int_t^v \sum_{k=1}^{d} \partial_2 \sigma^Y_{\cdot k}(u) \Phi_t(u) dW_k(u).$$

Thus, the (i,j) element, $i,j = 1,\ldots,p$, equals

$$\Phi_{ij,t}(v) = I_{ij} + \int_t^v \sum_{m=1}^{p} \frac{\partial \mu^Y_i(u)}{\partial Y_m} \Phi_{mj,t}(u) du + \int_t^v \sum_{k=1}^{d} \sum_{m=1}^{p} \frac{\partial \sigma^Y_{ik}(u)}{\partial Y_m} \Phi_{mj,t}(u) dW_k(u).$$

The Malliavin derivative of the first integral with respect to the n-th Brownian motion, $n = 1,\ldots,d$, equals

$$\begin{aligned}
&\mathcal{D}_{n,t}\left(\int_t^v \sum_{m=1}^{p} \frac{\partial \mu^Y_i(u)}{\partial Y_m} \Phi_{mj,t}(u) du\right) = \int_t^v \sum_{m=1}^{p} \left(\mathcal{D}_{n,t}\left(\frac{\partial \mu^Y_i(u)}{\partial Y_m}\right) \Phi_{mj,t}(u) + \frac{\partial \mu^Y_i(u)}{\partial Y_m} \mathcal{D}_{n,t}\left(\Phi_{mj,t}(u)\right)\right) du \\
&= \int_t^v \sum_{m=1}^{p} \left(\sum_{l=1}^{p} \frac{\partial^2 \mu^Y_i(u)}{\partial Y_m \partial Y_l} \mathcal{D}_{n,t}\left(Y_l(u)\right) \Phi_{mj,t}(u) + \frac{\partial \mu^Y_i(u)}{\partial Y_m} \mathcal{D}_{n,t}\left(\Phi_{mj,t}(u)\right)\right) du
\end{aligned}$$

The Malliavin derivative of the second integral equals

$$\begin{aligned}
&\mathcal{D}_{n,t}\left(\int_t^v \sum_{k=1}^{d} \sum_{m=1}^{p} \frac{\partial \sigma^Y_{ik}(u)}{\partial Y_m} \Phi_{mj,t}(u) dW_k(u)\right) \\
&= \sum_{m=1}^{p} \frac{\partial \sigma^Y_{in}(u)}{\partial Y_m} \Phi_{mj,t}(u) + \sum_{k=1}^{d} \sum_{m=1}^{p} \left(\mathcal{D}_{n,t}\left(\frac{\partial \sigma^Y_{ik}(u)}{\partial Y_m}\right) \Phi_{mj,t}(u) + \frac{\partial \sigma^Y_{ik}(u)}{\partial Y_m} \mathcal{D}_{n,t}\left(\Phi_{mj,t}(u)\right)\right) dW_k(u) \\
&= \sum_{m=1}^{p} \frac{\partial \sigma^Y_{in}(u)}{\partial Y_m} \Phi_{mj,t}(u) + \sum_{k=1}^{d} \sum_{m=1}^{p} \left(\sum_{l=1}^{p} \frac{\partial^2 \sigma^Y_{ik}(u)}{\partial Y_m \partial Y_l} \mathcal{D}_{n,t}\left(Y_l(u)\right) \Phi_{mj,t}(u) + \frac{\partial \sigma^Y_{ik}(u)}{\partial Y_m} \mathcal{D}_{n,t}\left(\Phi_{mj,t}(u)\right)\right) dW_k(u).
\end{aligned}$$

When equality (48),

$$\mathcal{D}_{j,t}\left(Y_l(v)\right) = \sum_{r=1}^{p} (\Phi_{lr,t} \sigma^Y_{rj}),$$

is substituted for $\mathcal{D}_{j,t}\left(Y_l(v)\right)$, the claim follows. The boundary condition comes from the constant first term of the second integral,

$$\sum_{m=1}^{p} \frac{\partial \sigma_{in}^{Y}(t)}{\partial Y_m} \Phi_{mj,t}(t) = \frac{\sigma_{in}^{Y}(t)}{\partial Y_j}.$$

$\square$

Proof of Lemma 2 . The three components equal zero for a relative risk aversion of 1 because for the myopic investor, all trading comes from the demand for the mean-variance efficient portfolio. Just as for the demand for the hedging portfolios, the expected volume due to hedging, EHVOLUME, changes sign for any given investment horizon at a risk aversion of 1. For the volume EHREBAL=$\mathbb{E}\left[d\pi_k^{(h)}(t)\right] w_j^{(h)}(t)$ from rebalancing the hedging portfolio, this follows from the fact that the term $\mathbb{E}\left[d\pi_k^{(h)}(t)\right]$ does not depend on the investor's risk aversion nor time horizon. Therefore, the change in sign results from the switch of sign of the demand $w_k^{(h)}(t)$ discussed above. Next, for the adjusting of the weight of the mean-variance portfolio $\pi_k^{(h)}(t)\left(\mathbb{E}\left[dw_j^{(h)}(t)\right] - w_j^{(h)}(t)\mu_k dt\right)$, the term $w_j^{(h)}(t)$ will display the same behavior. To understand the behavior of $\mathbb{E}\left[dw_j^{(h)}(t)\right]$, we see from (41) that

$$\mathbb{E}\left[dw_j^{(h)}(t)\right] = \left[w_{j,k}^{(h)}(t)\left(r(t) + \theta(t)'\theta(t)\right) - \rho\xi(t)^{-1}\theta(t)'\tilde{M}_{j,k}(t)\right] dt,$$

where $\tilde{M}_{j,k}(t)$ is defined in (42). The only effect of the risk aversion on $\tilde{M}_{j,k}(t)$ is via ρ. Thus, the second summand in $\mathbb{E}\left[dw_j^{(h)}(t)\right]$ has a constant sign for different values of the risk aversion, since the ρ cancels and the Malliavin derivative $\mathcal{D}_t\left(h_{k,t}(T)\right)$ is unaffected by risk aversion. Therefore, the behavior of the first summand, involving $w_{j,k}^{(h)}(t)$, determines the behavior of $\mathbb{E}\left[dw_j^{(h)}(t)\right]$. A similar analysis holds for $V(d\pi_k^{(h)}(t))V(dw_j^{(h)}(t))$ which equals, due to (45) and (41) and the fact that the stock's volatility $\Sigma(t)$ is assumed to be constant in this model,

$$V(d\pi_k^{(h)}(t))V(dw_j^{(h)}(t)) = \bar{\Sigma}_{ik}\sigma_{kj}^{\sigma^Y}\left[w_{j,k}^{(h)}(t)\theta(t)' - \rho\xi(t)^{-1}\tilde{M}_{j,k}(t)\right] dt.$$

It is again $w_{j,k}^{(h)}(t)$ in the first term that determines the sign of the expression for different values of the risk aversion. $\square$

Proof of Proposition 3 . Applying Ito's lemma to (63) yields on the left hand side

$$dD(t) = (A_0(t, D(t)) + A_1(t, D(t))\gamma_1(t))dt + B(t, D(t))d\nu_1(t). \tag{100}$$

Substituting on the right hand side $\xi_1(t)\eta(t)$ for $\xi_2(t)$ according to (61) leaves only the exogenous terms $\beta_i(t)$, $i = 1, 2$ and R as well as the endogenous quantities $y_i, i = 1, 2$, and $\xi_1(t)$. In the SDE, resulting from application of Ito's lemma, the drift term must equal the drift in (100) and the diffusion term must equal its counterpart in (100). These two equations can then be solved for $r(t)$ and $\theta_1(t)$. □

Proof of Lemma 3 . It follows from Ito's lemma that

$$\begin{aligned} d\triangle(t) &= d\gamma_2(t) - d\gamma_1(t) \\ &= a_1(t)\triangle(t)dt + \frac{A_1(t)}{B(t)}(V_2(t)d\nu_2(t) - V_1(t)d\nu_1(t)) \\ &= a_1(t)\triangle(t)dt + \frac{A_1(t)}{B(t)}\left(V_2(t)(d\nu_2(t) - d\nu_1(t)) + (V_2(t) - V_1(t))d\nu_1(t)\right). \end{aligned}$$

From the definition (58) of $\nu_i(t)$, it follow that

$$d\nu_2(t) - d\nu_1(t) = \frac{A_1(t)}{B(t)}(\gamma_1(t) - \gamma_2(t))dt = -\frac{A_1(t)}{B(t)}\triangle(t)dt$$

Hence,

$$d\triangle(t) = a_1(t)\triangle(t)dt - \left(\frac{A_1(t)}{B(t)}\right)^2 V_2(t)\triangle(t)dt + \frac{A_1(t)}{B(t)}(V_2(t) - V_1(t))d\nu_1(t).$$

□

B Parameters

The following parameters are used from Detemple et al. (2003, p. 414 and 441):

Parameter	Value
γ_r	1.1741
κ_r	0.0027668
$\bar{r}$	0.0063138×12
σ_r	$0.154055 \times 12^{1-1.1741}$
r_0	0.06
θ_0	0.10
σ	0.2
ϕ_r	$37.008/12^{2\times 0.45432}$
η_r	0.45432
κ_θ	0.85576
$\bar{\theta}$	0.048786
σ_θ	2.9417
θ_l	1.5
θ_u	1.5
$\gamma_{1,\theta}$	0.5
$\gamma_{2,\theta}$	2.8313
δ_θ	3.0708

C Tables

Table 1: Parameter values for the simulation of the incomplete information economy

Maximum likelihood estimates:	
$\hat{\bar{\gamma}}$	1.96% (0.0022)
$\hat{\phi}_\gamma$	1.16 (0.09)
$\hat{\sigma}_\gamma$	3.02% (0.0020)
Distribution of γ_0:	
$\hat{\gamma}_0$	1.96%
$\hat{V}_0$	0.02^2
Other parameters	
$\hat{\sigma}_D$	3.09%
Σ	0.2
D_0	1

The data used are from Shiller (2003), provided on Robert Shiller's web site `http://aida.econ.yale.edu/~shiller/data.htm`. From the U.S. real per capita consumption C_t from 1889 to 2003, the growth rates $\hat{\gamma}_t = \log(C_t) - \log(C_{t-1})$ were calculated. The first differences of $\hat{\gamma}_t$ were used for the maximum likelihood estimation. The number in brackets indicate the standard errors of the maximum likelihood estimates. The estimates for all three parameters are statistically significant at the 95% level. The mean of the distribution of γ_0 conditional on D_0 was set equal to the long-term growth rate $\hat{\bar{\gamma}}$. The variance $\hat{V}_0 = 0.02^2$ generates a reasonable range of initial values of γ_0. $\hat{\sigma}_D$ is the standard deviation of $\log(C_t) - \log(C_{t-1})$. The stock volatility Σ is chosen as constant and set equal to the historical average for the U.S. stock market (Detemple et al., 2003). The level of the aggregate endowment is normalized to 1.

Table 2: Expected volume

T	EVOLUME	EMVVOLUME	EMVTRAD	EMVREBAL	COVMV	EHVOLUME	EHTRAD	EHREBAL	COVH
				R=0.75					
1	-0.238	-0.225	-0.004	-0.302	0.081	-0.013	0.015	0.001	-0.029
4	-0.239	-0.226	-0.005	-0.302	0.080	-0.013	0.016	0.001	-0.030
7	-0.242	-0.229	-0.005	-0.302	0.078	-0.013	0.017	0.001	-0.031
10	-0.244	-0.231	-0.005	-0.302	0.075	-0.013	0.017	0.001	-0.031
				R=3					
T	EVOLUME	EMVVOLUME	EMVTRAD	EMVREBAL	COVMV	EHVOLUME	EHTRAD	EHREBAL	COVH
1	-0.047	-0.073	-0.003	-0.075	0.005	0.026	-0.030	-0.001	0.056
4	-0.048	-0.072	-0.003	-0.075	0.006	0.025	-0.030	-0.002	0.057
7	-0.047	-0.071	-0.003	-0.075	0.007	0.024	-0.032	-0.002	0.058
10	-0.046	-0.070	-0.003	-0.075	0.008	0.024	-0.033	-0.002	0.059

Total expected volume EVOLUME is split into its two components according to (56), one from the mean-variance portfolio,

$$\begin{aligned} \text{EMVVOLUME} =& \text{EMVREBAL} && + \text{EMVTRAD} && + \text{COVMV} \\ =& \frac{1}{R} X_i(t)\, \mathbb{E}\left[d\pi_k^{(mv)}(t)\right] && + \frac{1}{R}\pi_k^{(mv)}(t)(\mathbb{E}\left[dX_i(t)\right] - X_i(t)\mu_k(t)dt) && + \frac{1}{R} V(d\pi_k^{(mv)}(t))V(dX_i(t))), \end{aligned}$$

and one from the hedging portfolio,

$$\begin{aligned} \text{EHVOLUME} =& \text{EHREBAL} && + \text{EHTRAD} && + \text{COVH} \\ =& \mathbb{E}\left[d\pi_k^{(h)}(t)\right] w_i^{(h)}(t) && + \pi_k^{(h)}(t)\left(\mathbb{E}\left[dw_i^{(h)}(t)\right] - w_i^{(h)}(t)\mu_k dt\right) && + V(d\pi_k^{(h)}(t))V(dw_i^{(h)}(t)). \end{aligned}$$

Table 3: Volatility of volume

	T	VVOLUME	VMVVOLUME	VMVTRAD	VMVREBAL	VHVOLUME	VHTRAD	VHREBAL
R=0.75	1	0.693	0.541	-0.066	0.607	0.153	0.154	-0.001
	4	0.700	0.540	-0.068	0.607	0.160	0.161	-0.001
	7	0.706	0.538	-0.069	0.607	0.168	0.167	0.001
	10	0.712	0.536	-0.071	0.607	0.176	0.174	0.002
R=1	1	0.405	0.405	-0.050	0.455	0.000	0.000	0.000
	4	0.405	0.405	-0.050	0.455	0.000	0.000	0.000
	7	0.405	0.405	-0.050	0.455	0.000	0.000	0.000
	10	0.405	0.405	-0.050	0.455	0.000	0.000	0.000
R=3	1	-0.161	0.136	-0.016	0.152	-0.296	-0.296	0.000
	4	-0.166	0.138	-0.014	0.152	-0.305	-0.304	-0.001
	7	-0.177	0.141	-0.011	0.152	-0.317	-0.316	-0.002
	10	-0.186	0.144	-0.008	0.152	-0.330	-0.326	-0.004

Total volatility of volume VVOLUME is split into its two components according to (56), one from the mean-variance portfolio,

$$\begin{aligned} \text{VMVVOLUME} =& \text{VMVTRAD} && + \text{VMVREBAL} \\ =& \frac{1}{R}\pi_k^{(mv)}(t)(V(dX_i(t)) - X_i t V(dG_k(t))) && + \frac{1}{R}X_i(t)V(d\pi_k^{(mv)}(t)), \end{aligned}$$

and one from the hedging portfolio,

$$\begin{aligned} \text{VHVOLUME} =& \text{VHTRAD} && + \text{VHREBAL} \\ =& \pi_k^{(h)}(t)(V(dw_i^{(h)}(t)) - w_i^{(h)}(t)'V(dG_k(t))) && + w_i^{(h)}(t)V(d\pi^{(h)}(t)). \end{aligned}$$

Table 4: Expected volume, $R = 0.75$

		$\triangle(0)$	*Total* T				*Mean-variance component* T				*Hedging component* T			
			1	4	7	10	1	4	7	10	1	4	7	10
$V_0^{(2)} = 0.01^2$	$\delta = 0$	-0.025	2.07	6.54	6.83	6.89	0.77	0.05	0.00	0.00	1.29	6.48	6.83	6.89
		-0.010	0.15	1.98	2.05	2.07	−0.02	0.04	0.00	0.00	0.17	1.94	2.05	2.07
		0.000	3.08	0.01	−0.02	−0.02	2.95	0.02	0.00	0.00	0.13	−0.01	−0.02	−0.02
		0.010	3.69	−3.01	−3.22	−3.28	5.36	0.01	0.00	0.00	−1.67	−3.02	−3.22	−3.28
		0.025	−10.25	−13.00	−13.86	−14.11	1.55	−0.01	0.00	0.00	−11.81	−12.99	−13.85	−14.10
	$\delta = 0.1$	-0.025	−2.37	8.20	8.87	9.08	−0.96	−0.01	0.00	0.00	−1.41	8.22	8.87	9.09
		-0.010	9.77	5.20	5.67	5.80	8.53	−0.06	0.00	0.00	1.24	5.26	5.68	5.81
		0.000	14.10	−0.63	−0.54	−0.54	13.68	−0.10	−0.01	−0.01	0.43	−0.53	−0.53	−0.54
		0.010	9.58	−11.01	−11.60	−11.81	14.28	−0.16	−0.01	−0.01	−4.70	−10.85	−11.59	−11.80
		0.025	−22.26	−37.03	−39.63	−40.25	0.81	−0.23	−0.01	−0.01	−23.07	−36.80	−39.62	−40.24
$V_0^{(2)} = 0.05^2$	$\delta = 0$	-0.025	3.06	0.77	0.66	0.64	0.84	0.03	0.00	0.00	2.22	0.74	0.66	0.64
		-0.010	0.17	−0.10	−0.12	−0.13	0.29	0.01	0.00	0.00	−0.12	−0.11	−0.12	−0.13
		0.000	−0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	−0.01	0.00	0.00	0.00
		0.010	−0.93	−0.44	−0.43	−0.44	−0.25	−0.01	0.00	0.00	−0.68	−0.42	−0.43	−0.44
		0.025	−7.61	−4.08	−4.08	−4.15	−0.61	−0.03	0.00	0.00	−7.00	−4.05	−4.08	−4.14
	$\delta = 0.1$	-0.025	−1.01	−1.42	−1.51	−1.56	1.00	0.03	0.00	0.00	−2.01	−1.45	−1.51	−1.56
		-0.010	−0.50	−0.44	−0.46	−0.47	0.30	0.01	0.00	0.00	−0.80	−0.45	−0.46	−0.47
		0.000	−0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	−0.01	0.00	0.00	0.00
		0.010	−1.65	−0.83	−0.82	−0.84	−0.25	−0.01	0.00	0.00	−1.40	−0.82	−0.82	−0.84
		0.025	−11.81	−6.37	−6.35	−6.44	−0.55	−0.03	0.00	0.00	−11.26	−6.34	−6.35	−6.43

$V_0^{(1)} = 0.05^2$ is kept constant.

$\delta = \gamma_0^{(1)} - \hat{\gamma}_0$ denotes the error of the reference investor with respect to the true moment of γ_0.

Table 5: Volatility of volume, $R = 0.75$

		$\triangle(0)$	*Total* T				*Mean-variance component* T				*Hedging component* T			
			1	4	7	10	1	4	7	10	1	4	7	10
$V_0^{(2)} = 0.01^2$	$\delta = 0$	-0.025	7.67	-1.14	-0.40	-0.39	12.09	-0.28	-0.08	-0.08	-4.42	-0.85	-0.32	-0.31
		-0.01	-4.94	-0.31	0.04	0.05	-0.56	-0.22	-0.06	-0.06	-4.38	-0.09	0.10	0.11
		0	-0.72	-0.05	0.16	0.17	-1.21	-0.21	-0.06	-0.06	0.50	0.16	0.22	0.24
		0.01	14.09	0.07	0.18	0.20	4.69	-0.16	-0.08	-0.08	9.40	0.23	0.25	0.28
		0.025	54.33	-0.51	-0.55	-0.52	25.33	0.04	-0.12	-0.13	29.00	-0.56	-0.43	-0.40
	$\delta = 0.1$	-0.025	-34.60	-0.37	-0.48	-0.47	-20.38	-0.01	-0.08	-0.09	-14.22	-0.36	-0.39	-0.38
		-0.01	-22.44	0.48	0.37	0.41	-15.26	0.25	-0.13	-0.13	-7.19	0.23	0.50	0.55
		0	-1.42	0.71	0.83	0.90	-3.63	0.44	-0.16	-0.18	2.21	0.27	1.00	1.08
		0.01	28.31	0.11	0.79	0.86	13.96	0.60	-0.21	-0.23	14.35	-0.49	1.00	1.10
		0.025	83.57	-3.98	-1.50	-1.44	49.32	0.61	-0.30	-0.33	34.25	-4.58	-1.21	-1.11
$V_0^{(2)} = 0.05^2$	$\delta = 0$	-0.025	0.38	-0.15	-0.12	-0.12	0.44	-0.05	-0.03	-0.03	-0.05	-0.11	-0.10	-0.09
		-0.01	0.05	-0.12	-0.12	-0.12	0.19	-0.02	-0.01	-0.01	-0.14	-0.10	-0.10	-0.11
		0	-0.32	-0.11	-0.11	-0.11	0.02	-0.01	-0.01	-0.01	-0.33	-0.10	-0.09	-0.10
		0.01	-0.55	-0.09	-0.08	-0.08	0.00	-0.01	-0.02	-0.02	-0.55	-0.08	-0.05	-0.05
		0.025	-1.58	-0.30	-0.26	-0.25	0.02	0.01	-0.05	-0.05	-1.60	-0.31	-0.21	-0.20
	$\delta = 0.1$	-0.025	1.10	-0.02	-0.01	-0.02	0.87	-0.01	-0.01	-0.01	0.23	-0.01	0.00	0.00
		-0.01	0.08	-0.13	-0.13	-0.14	0.23	-0.01	-0.01	-0.01	-0.15	-0.12	-0.13	-0.13
		0	-0.32	-0.11	-0.11	-0.11	0.02	-0.01	-0.01	-0.01	-0.33	-0.10	-0.09	-0.10
		0.01	-0.56	-0.08	-0.06	-0.06	0.00	-0.01	-0.03	-0.03	-0.57	-0.07	-0.03	-0.03
		0.025	-2.25	-0.42	-0.36	-0.35	-0.15	0.02	-0.06	-0.06	-2.09	-0.44	-0.30	-0.29

$V_0^{(1)} = 0.05^2$ is kept constant.
$\delta = \gamma_0^{(1)} - \hat{\gamma}_0$ denotes the error of the reference investor with respect to the true moment of γ_0.

Table 6: Expected volume, $R = 3$

		$\triangle(0)$	*Total* T 1	4	7	10	*Mean-variance component* T 1	4	7	10	*Hedging component* T 1	4	7	10
$V_0^{(2)} = 0.01^2$	$\delta = 0$	-0.025	-0.47	-0.42	-0.38	-0.37	-0.01	-0.01	-0.01	-0.01	-0.46	-0.41	-0.36	-0.35
		-0.010	-0.27	-0.31	-0.31	-0.31	-0.01	-0.01	-0.01	-0.01	-0.26	-0.30	-0.29	-0.29
		0.000	0.02	0.03	0.04	0.04	-0.01	-0.01	-0.01	-0.01	0.02	0.04	0.05	0.05
		0.010	0.43	0.58	0.61	0.62	-0.01	-0.01	-0.01	-0.01	0.44	0.58	0.62	0.63
		0.025	1.26	1.76	1.89	1.93	0.00	-0.01	-0.01	-0.01	1.27	1.77	1.90	1.94
	$\delta = 0.1$	-0.025	-0.73	-1.25	-1.34	-1.38	-0.01	-0.01	-0.01	-0.01	-0.72	-1.25	-1.33	-1.37
		-0.010	-0.50	-0.79	-0.83	-0.85	0.00	-0.02	-0.01	-0.01	-0.50	-0.78	-0.82	-0.84
		0.000	0.19	0.27	0.32	0.33	0.00	-0.02	0.00	0.00	0.19	0.29	0.32	0.33
		0.010	1.28	1.90	2.08	2.14	0.00	-0.03	0.00	0.00	1.28	1.92	2.09	2.14
		0.025	3.61	5.29	5.81	5.96	0.00	-0.04	0.00	0.00	3.61	5.33	5.81	5.95
$V_0^{(2)} = 0.05^2$	$\delta = 0$	-0.025	0.32	0.53	0.55	0.57	-0.01	-0.02	-0.02	-0.02	0.33	0.55	0.57	0.59
		-0.010	0.06	0.11	0.11	0.12	-0.01	-0.02	-0.02	-0.02	0.07	0.13	0.13	0.13
		0.000	0.01	0.03	0.03	0.03	-0.01	-0.02	-0.01	-0.02	0.02	0.05	0.04	0.05
		0.010	0.06	0.11	0.12	0.12	-0.01	-0.01	-0.01	-0.01	0.07	0.13	0.13	0.13
		0.025	0.32	0.54	0.57	0.58	-0.01	-0.01	-0.01	-0.01	0.33	0.56	0.58	0.59
	$\delta = 0.1$	-0.025	1.11	1.52	1.54	1.56	-0.01	-0.03	-0.02	-0.02	1.12	1.55	1.56	1.58
		-0.010	0.17	0.24	0.24	0.24	-0.01	-0.02	-0.02	-0.02	0.17	0.26	0.26	0.26
		0.000	0.01	0.03	0.03	0.03	-0.01	-0.02	-0.01	-0.02	0.02	0.05	0.04	0.05
		0.010	0.24	0.34	0.34	0.35	-0.01	-0.01	-0.01	-0.01	0.25	0.35	0.36	0.36
		0.025	1.26	1.75	1.78	1.81	0.00	-0.01	-0.01	-0.01	1.27	1.77	1.79	1.82

$V_0^{(1)} = 0.05^2$ is kept constant.
$\delta = \gamma_0^{(1)} - \hat{\gamma}_0$ denotes the error of the reference investor with respect to the true moment of γ_0.

Table 7: Volatility of volume, $R = 3$

			Total				*Mean-variance component*				*Hedging component*			
		$\triangle(0)$	T				T				T			
			1	4	7	10	1	4	7	10	1	4	7	10
$V_0^{(2)} = 0.01^2$	$\delta = 0$	-0.025	0.08	−0.41	0.10	0.11	−0.16	−0.49	−0.38	−0.39	0.24	0.09	0.48	0.50
		-0.01	0.03	−0.21	0.07	0.07	−0.14	−0.35	−0.33	−0.34	0.17	0.14	0.40	0.42
		0	0.01	−0.10	0.04	0.05	−0.12	−0.27	−0.30	−0.31	0.13	0.17	0.35	0.36
		0.01	0.01	0.03	0.05	0.05	−0.11	−0.20	−0.27	−0.28	0.12	0.23	0.32	0.33
		0.025	0.08	0.28	0.17	0.17	−0.08	−0.12	−0.22	−0.23	0.16	0.40	0.39	0.40
	$\delta = 0.1$	-0.025	0.08	0.08	0.17	0.18	−0.12	0.04	−0.26	−0.26	0.20	0.05	0.43	0.44
		-0.01	0.02	−0.05	0.07	0.07	−0.05	0.04	−0.14	−0.14	0.07	−0.09	0.21	0.21
		0	−0.01	−0.14	0.03	0.03	0.00	0.01	−0.06	−0.05	−0.01	−0.15	0.09	0.08
		0.01	0.02	−0.14	0.08	0.08	0.05	−0.02	0.01	0.03	−0.04	−0.13	0.06	0.05
		0.025	0.24	0.16	0.45	0.46	0.13	−0.07	0.13	0.15	0.12	0.23	0.33	0.31
$V_0^{(2)} = 0.05^2$	$\delta = 0$	-0.025	0.02	−0.27	0.04	0.04	−0.24	−0.63	−0.47	−0.49	0.26	0.36	0.51	0.53
		-0.01	0.03	−0.05	0.06	0.07	−0.22	−0.51	−0.44	−0.46	0.25	0.46	0.50	0.52
		0	0.02	0.06	0.05	0.06	−0.21	−0.43	−0.42	−0.43	0.22	0.49	0.47	0.49
		0.01	0.01	0.16	0.04	0.05	−0.19	−0.36	−0.39	−0.41	0.20	0.52	0.44	0.45
		0.025	0.01	0.33	0.07	0.07	−0.17	−0.26	−0.36	−0.37	0.19	0.59	0.43	0.44
	$\delta = 0.1$	-0.025	−0.06	−0.43	−0.06	−0.06	−0.32	−0.76	−0.58	−0.60	0.26	0.33	0.52	0.55
		-0.01	0.02	−0.07	0.05	0.06	−0.25	−0.55	−0.48	−0.50	0.27	0.48	0.53	0.56
		0	0.02	0.06	0.05	0.06	−0.21	−0.43	−0.42	−0.43	0.22	0.49	0.47	0.49
		0.01	0.01	0.16	0.05	0.05	−0.16	−0.33	−0.35	−0.36	0.17	0.49	0.40	0.42
		0.025	0.09	0.38	0.16	0.17	−0.09	−0.21	−0.26	−0.26	0.18	0.59	0.42	0.43

$V_0^{(1)} = 0.05^2$ is kept constant.

$\delta = \gamma_0^{(1)} - \hat{\gamma}_0$ denotes the error of the reference investor with respect to the true moment of γ_0.

D Figures

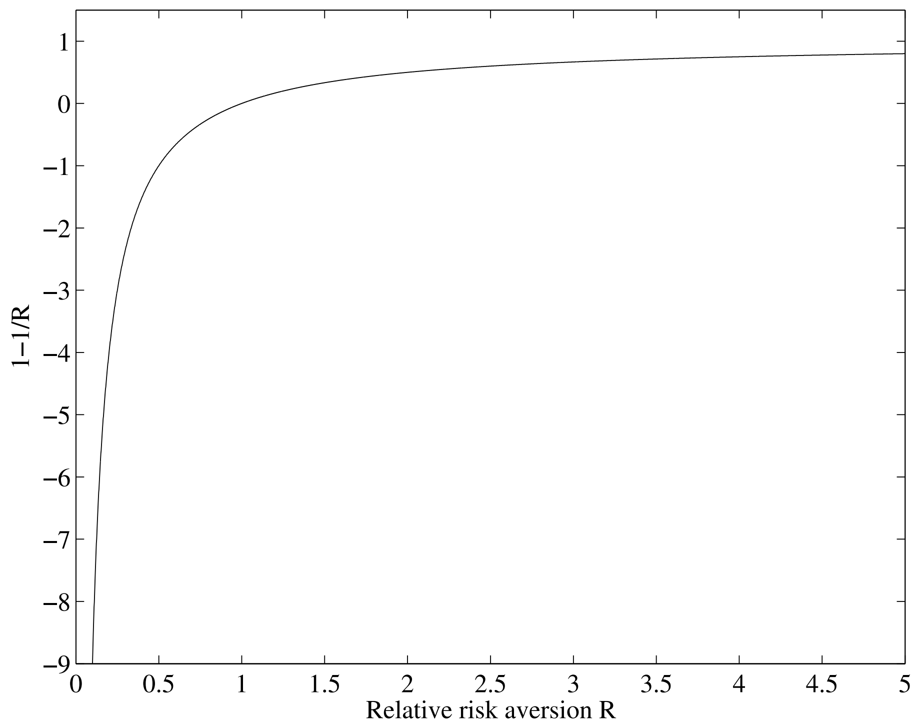
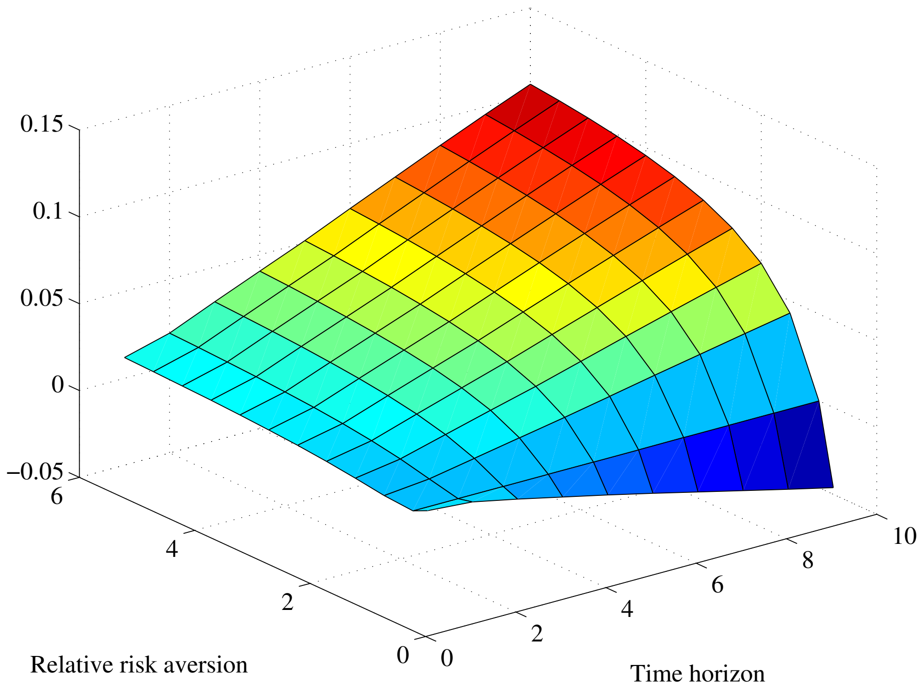
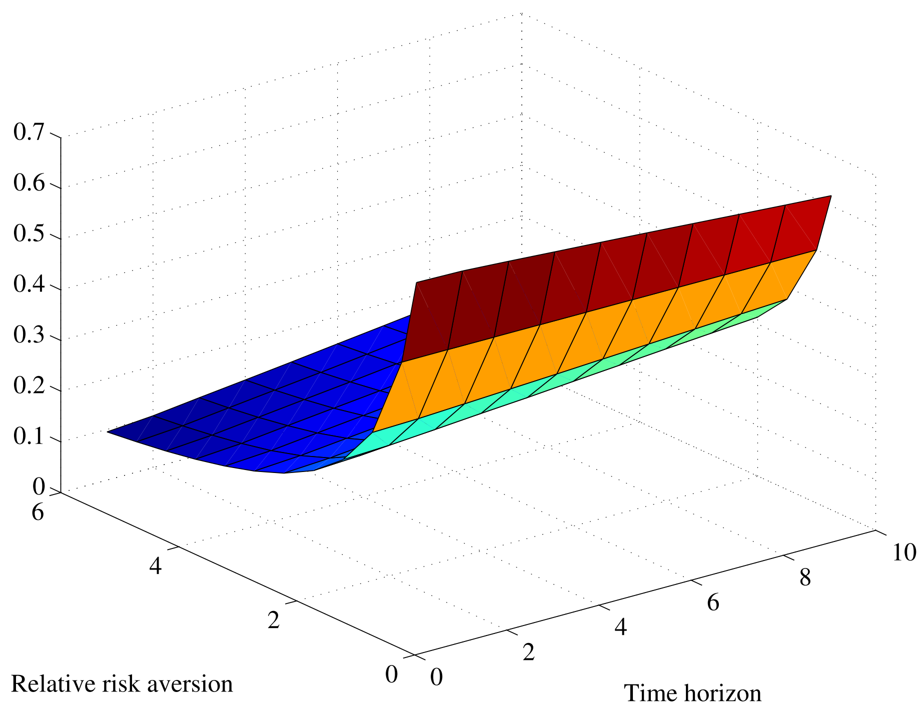


Figure 1: Impact of the relative risk aversion on the weighting of $h_t(v)$

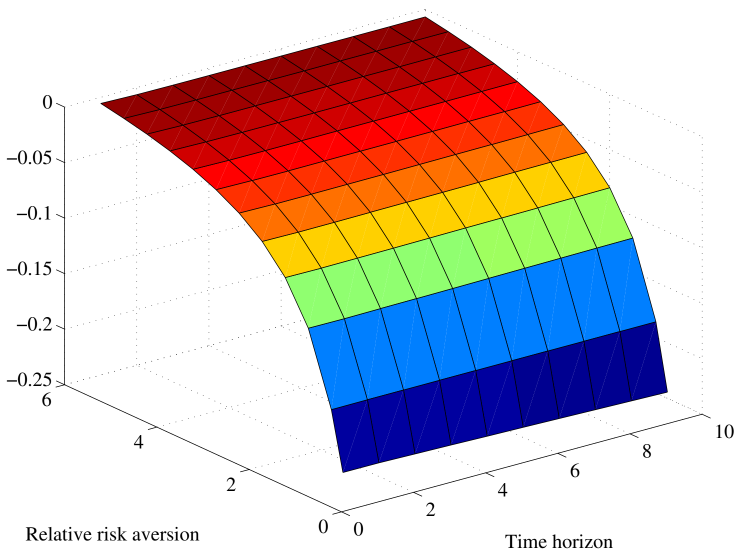


(a) Total hedging demand $\pi^{(h)}(0)w_i^{(h)}(0)$

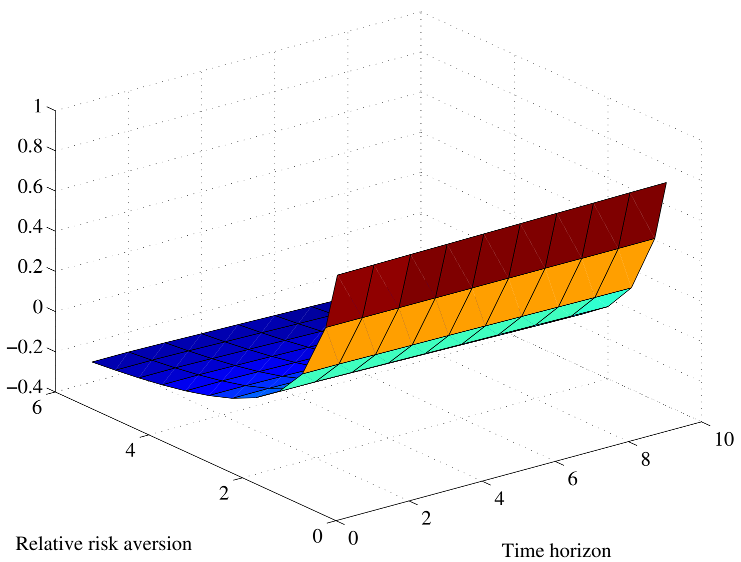


(b) Total amount invested in the stock, $\pi_i(0)$

Figure 2: Amounts invested in the stock for the investment problem



(a) Expected trading volume



(b) Volatility of trading volume

Figure 3: Moments of trading volume for the investment problem